SLMath Summer School
Isogeny-based cryptography
Day 2

Chloe Martindale

University of Bristol

Recall: Diffie–Hellman key exchange '76

Public parameters:

- a prime p (experts: uses $\mathbb{F}_p^*$, today also elliptic curves)
- a number $g \pmod{p}$ (nonexperts: think of an integer less than p)

Alice	Eve	Bob
$\mathrm{sk}_A \xleftarrow{\text{random}} \{0 \ldots p-1\}$		$\mathrm{sk}_B \xleftarrow{\text{random}} \{0 \ldots p-1\}$
$\mathrm{pk}_A = g^{\mathrm{sk}_A}$		$\mathrm{pk}_B = g^{\mathrm{sk}_B}$
$ss := (g^{\mathrm{sk}_B})^{\mathrm{sk}_A}$		$ss := (g^{\mathrm{sk}_A})^{\mathrm{sk}_B}$

- Alice and Bob agree on a shared secret key ss, then they can use that to encrypt their messages.
- Eve sees $\mathrm{pk}_A = g^{\mathrm{sk}_A}$, $\mathrm{pk}_B = g^{\mathrm{sk}_B}$; can't find sk_A, sk_B, ss.

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Broken by Shor!

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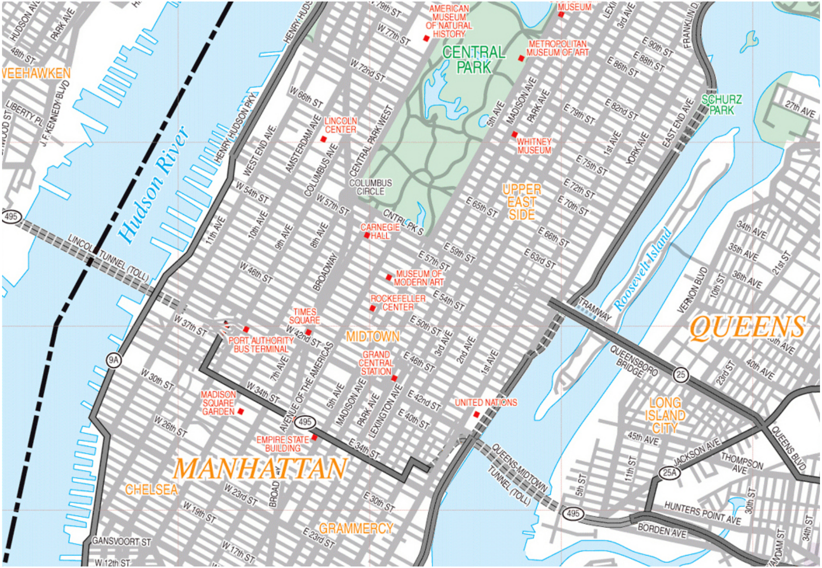
Graph walking Diffie–Hellman?



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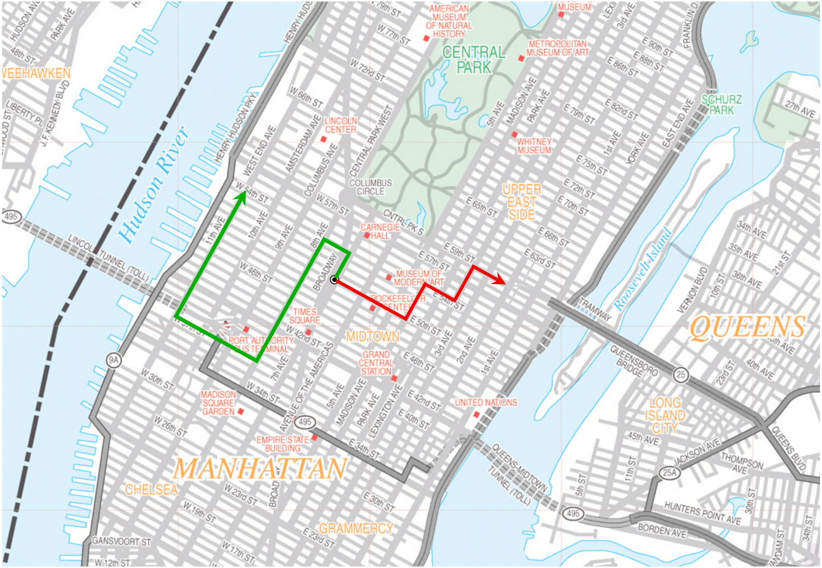

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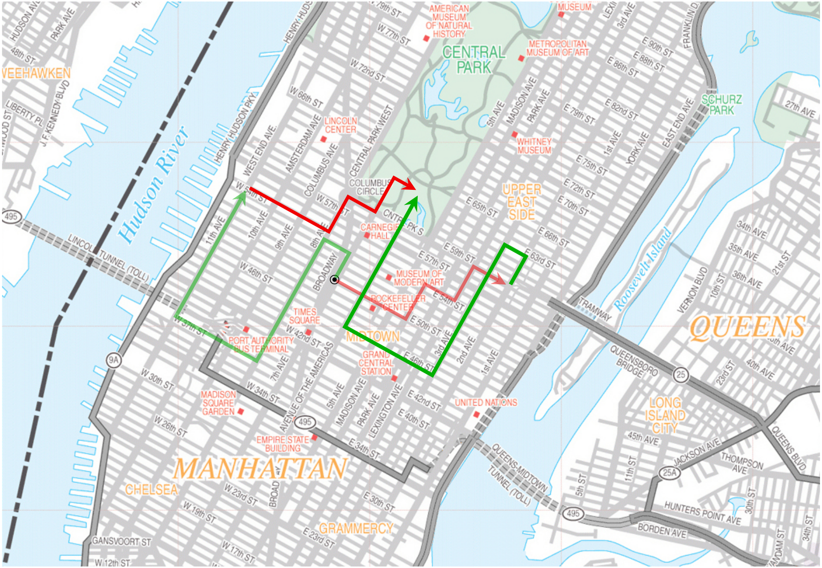

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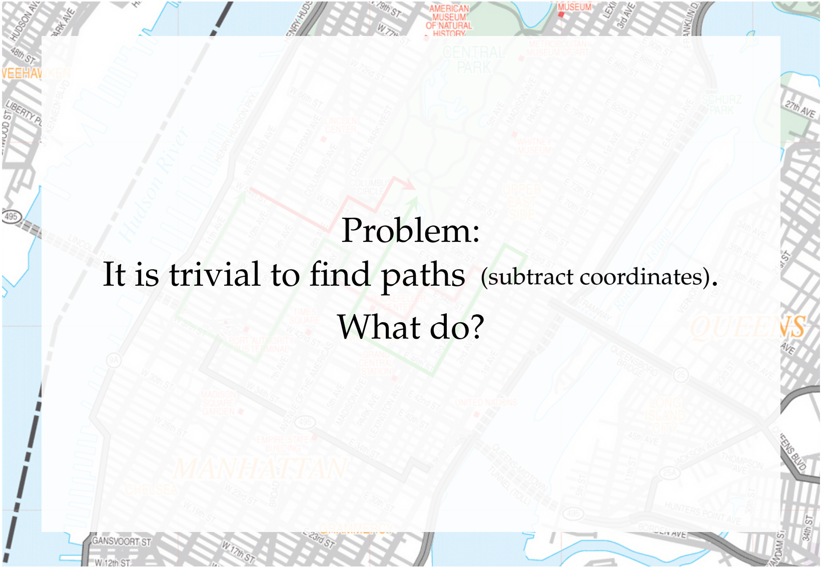

Problem:

It is trivial to find paths (subtract coordinates).

What do?

Big picture 🔍

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It is easy to construct graphs that satisfy *almost* all of these — not enough for crypto!

Stand back!

We're going to do maths.

Maths background #1/3: Isogenies *(edges)*

An isogeny of elliptic curves is a non-zero map $E \to E'$ that is:

- given by rational functions.
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Each isogeny $\varphi\colon E \to E'$ has a unique dual isogeny $\widehat{\varphi}\colon E' \to E$ characterized by $\widehat{\varphi} \circ \varphi = \varphi \circ \widehat{\varphi} = [\deg \varphi]$.

Maths background #2/3: Isogenies and kernels

For any finite subgroup G of E, there exists a unique[1] separable isogeny $\varphi_G \colon E \to E'$ with kernel G.

The curve E' is denoted by E/G. (cf. quotient groups)

If G is defined over k, then φ_G and E/G are also defined over k.

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Vélu operates in the field where the points in G live.
$\rightsquigarrow$ need to make sure extensions stay small for desired $\#G$
$\rightsquigarrow$ this is why we use supersingular curves!

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Math slide #3/3: Supersingular isogeny graphs

Let p be a prime, q a power of p, and ℓ a positive integer $\notin p\mathbb{Z}$.

An elliptic curve $E/\mathbb{F}_q$ is *supersingular* if $p \mid (q + 1 - \#E(\mathbb{F}_q))$.

We care about the cases $\#E(\mathbb{F}_p) = p + 1$ and $\#E(\mathbb{F}_{p^2}) = (p + 1)^2$.

$\rightsquigarrow$ easy way to control the group structure by choosing p!

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Let $S \not\ni p$ denote a set of prime numbers.

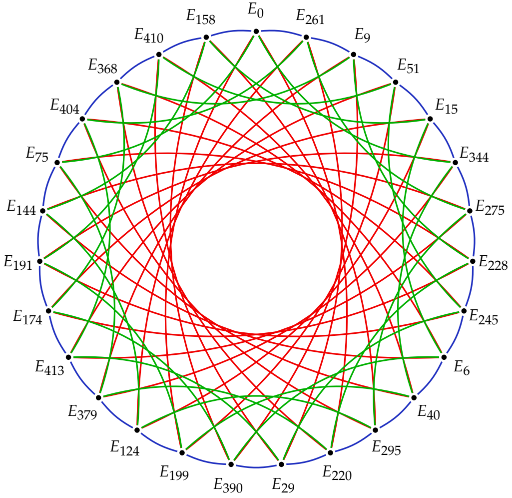
The supersingular S-isogeny graph over $\mathbb{F}_q$ consists of:

- vertices given by isomorphism classes of supersingular elliptic curves,
- edges given by equivalence classes[1] of ℓ-isogenies ($\ell \in S$),

both defined over $\mathbb{F}_q$.

[1] Two isogenies $\varphi\colon E \to E'$ and $\psi\colon E \to E''$ are identified if $\psi = \iota \circ \varphi$ for some isomorphism $\iota\colon E' \to E''$.

Graphs of elliptic curves



Nodes: Supersingular curves $E_A: y^2 = x^3 + Ax^2 + x$ over $\mathbb{F}_{419}$.
Edges: 3-, 5-, and 7-isogenies

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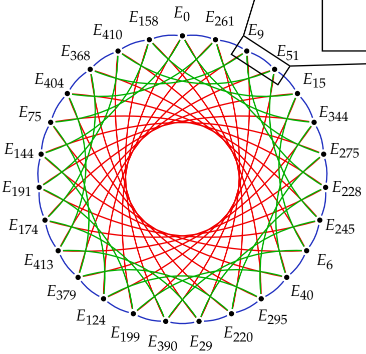
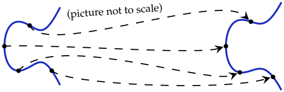


A 3-isogeny



$$E_{51}\colon y^2=x^3+51x^2+x \longrightarrow E_9\colon y^2=x^3+9x^2+x$$

$$(x,y) \longmapsto \left(\frac{97x^3-183x^2+x}{x^2-183x+97},\; y\cdot\frac{133x^3+154x^2-5x+97}{-x^3+65x^2+128x-133}\right)$$

[ˈsiːˌsaɪd]

CRS or CSIDH

Traditionally, Diffie-Hellman works in a group G via the map

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$\rightsquigarrow$ Idea:

Replace exponentiation on the group G by a group action of a group H on a set S:

$$H \times S \to S.$$

Quantumifying Exponentiation

- We want to replace the exponentiation map

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- The action of a well-chosen $\mathfrak{l} \in \mathrm{cl}(\mathbb{Z}[\sqrt{-p}])$ on S moves the elliptic curves one step around one of the cycles.

$$\begin{array}{ccc} \mathrm{cl}(\mathbb{Z}[\sqrt{-p}]) \times S & \to & S \\ (\mathfrak{l}_3, E) & \mapsto & \mathfrak{l}_3 * E. \end{array}$$

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Diffie and Hellman go to the CSIDH

Alice

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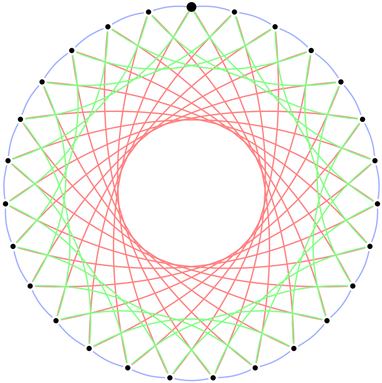

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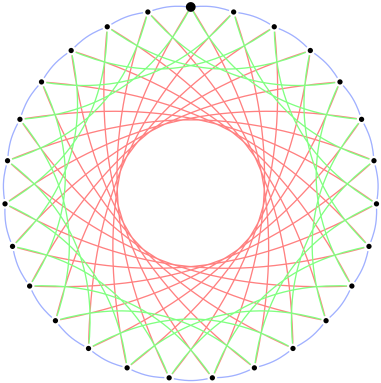

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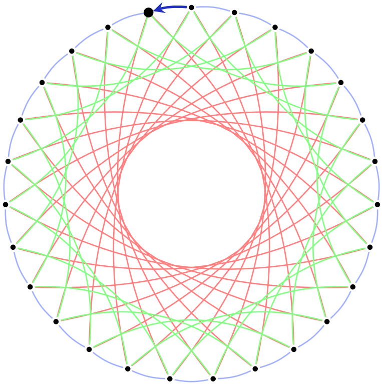

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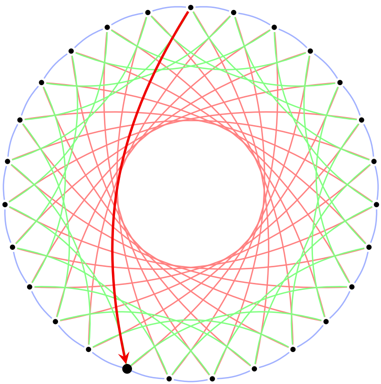

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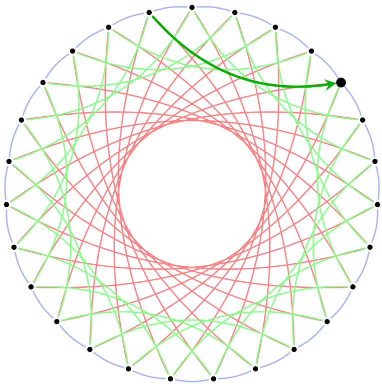

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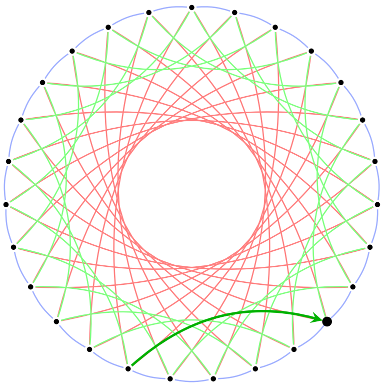

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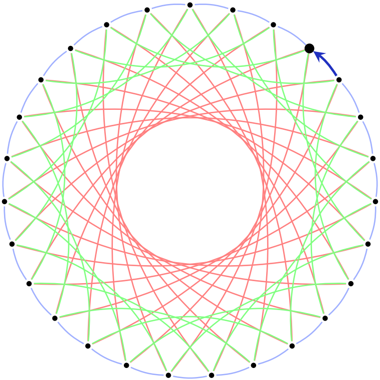

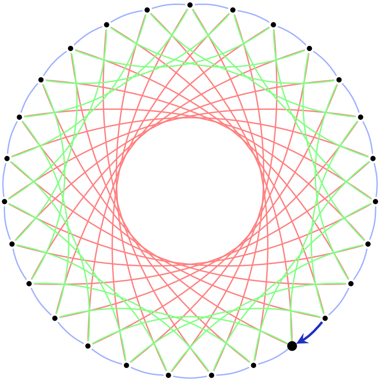

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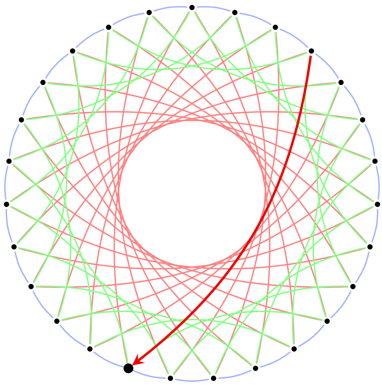

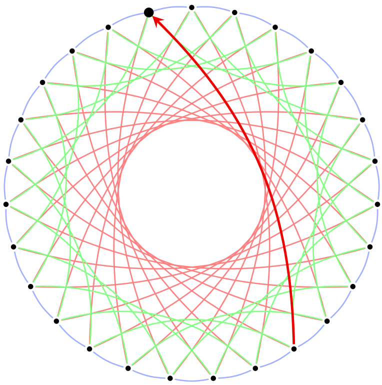

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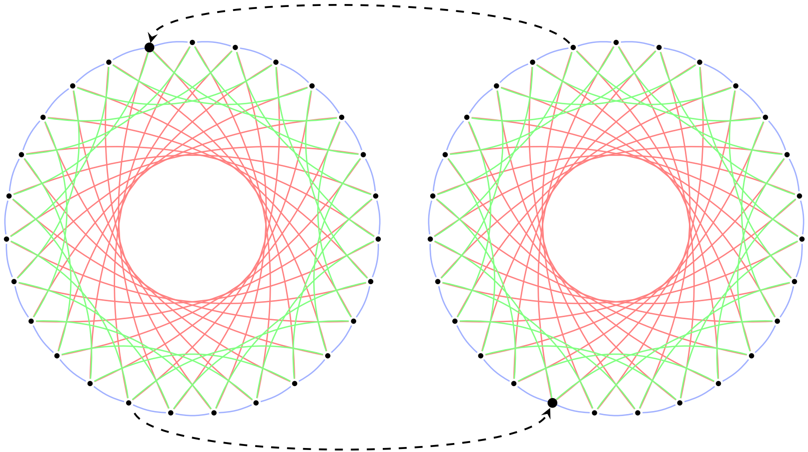

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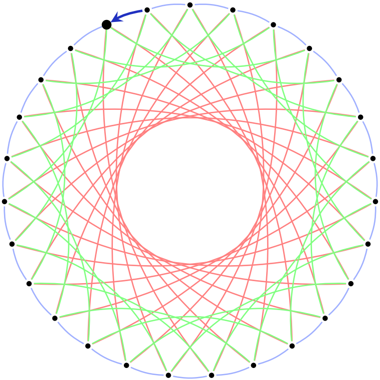

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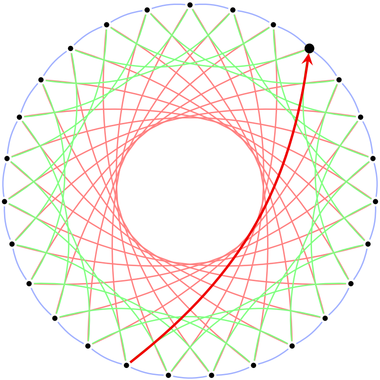

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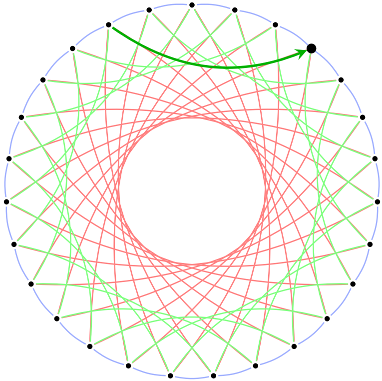

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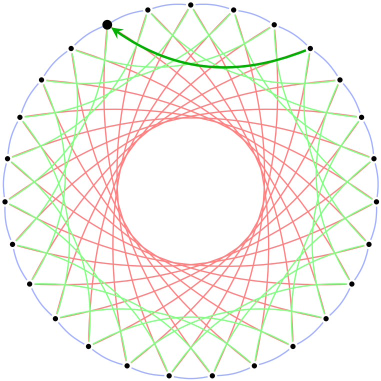

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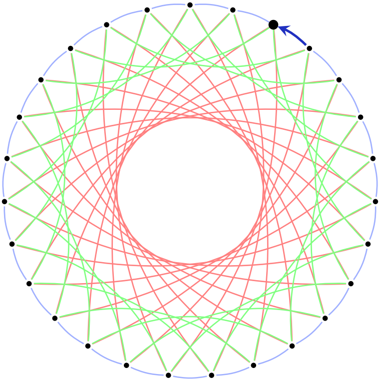

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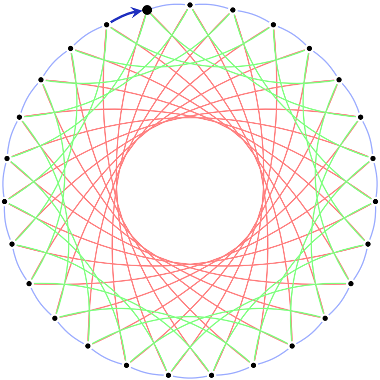

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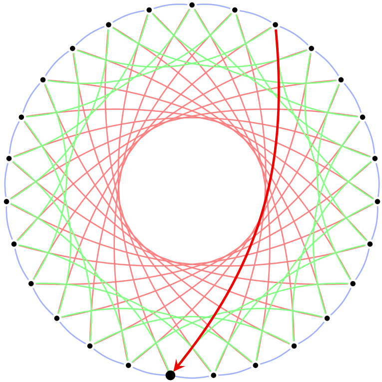

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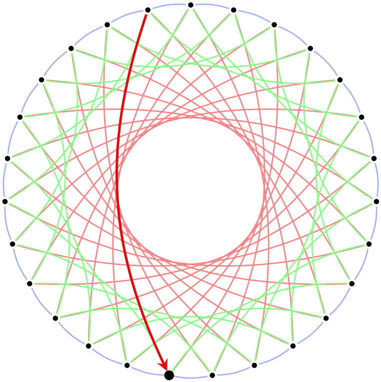

Diffie and Hellman go to the CSIDH

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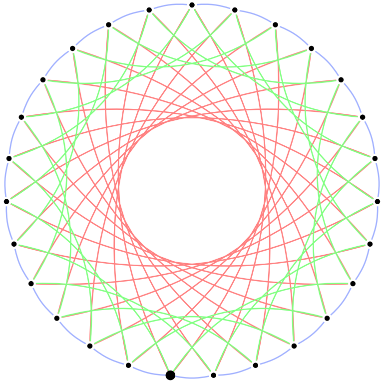

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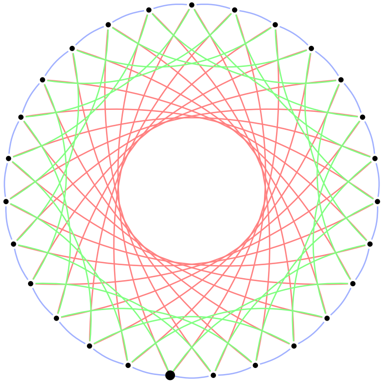

Choosing parameters

In [CLMPR18], parameters are chosen as follows:

- $\ell_1, \ldots, \ell_{n-1}$ the first $n-1$ odd primes.
- $\ell_n > \ell_{n-1}$ the smallest prime such that $p = 4\ell_1 \cdots \ell_n - 1$ is prime.

Then:

- $\mathfrak{l}_1, \ldots, \mathfrak{l}_n$ correspond to kernels of $\mathbb{F}_p$-rational isogenies (see next slide) — fast.
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*Any $I \in \mathrm{cl}(\mathbb{Z}[\sqrt{-p}])$ can be written as $\prod \mathfrak{l}_i^{e_i}$ with $e_i \in [-5, \ldots, 5]$.

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To compute a neighbour of E, we have to compute an ℓ-isogeny from E. To do this:

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- About $\sqrt{p}$ of all $A \in \mathbb{F}_p$ are valid keys.
- Public-key validation: Check that E_A has $p + 1$ points.

  Easy Monte-Carlo algorithm: Pick random P on E_A and check $[p+1]P = \infty$.[1]

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- Part of CJS attack computes many paths in superposition.

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Original proposal in 2018 paper: $\mathbb{F}_p \approx 512$ bits.

- The exact cost of the Kuperberg/Regev/CJS attack is subtle – it depends on:
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- Overheads from error correction, high quantum memory etc., not yet understood.

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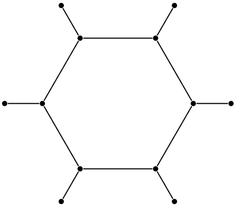
- The 2-isogeny graph looks like this:



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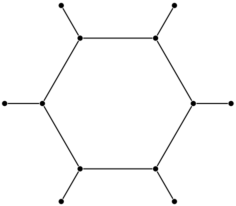

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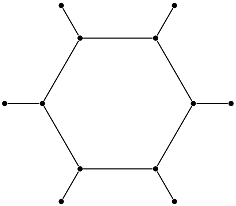

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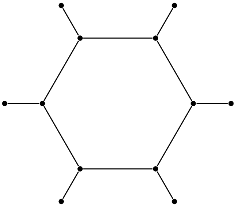

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$\rightsquigarrow$ How to compute 'on the surface'?

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- For any curve on the surface, the 2-isogeny with kernel $\langle(0, 0)\rangle$ is horizontal.

Venturing further beyond the CSIDH

A selection of more advances since original publication (2018):

- sqrtVelu [BDLS20]: square-root speed-up on computation of large-degree isogenies.
- Radical isogenies [CDV20]: significant speed-up on isogenies of small-ish degree.
- Some work on different curve forms (e.g. Edwards).
- Knowledge of $\mathrm{End}(E_0)$ and $\mathrm{End}(E_A)$ breaks CSIDH in classical polynomial time [Wes21].
- CTIDH [$\mathrm{B}^2\mathrm{C}^2\mathrm{LMS}^2$]: Efficient constant-time CSIDH-style construction.

What about signatures? (S '06, DG '18, BKV '19, DFKLMPW '23)

Identification protocol:

- Alice generates $(\mathsf{sk}_A, \mathsf{pk}_A)$, publishes pk_A.
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4. Verifier: P, pk, $\mathsf{epk} \rightsquigarrow$ valid (or not!)

Identification scheme from $H \times S \to S$

Prover	**Public**	**Verifier**
	$E \in S, \mathfrak{l}_i \in H$	
$s_i \xleftarrow{\$} \mathbb{Z}$		
$\mathsf{sk} = \prod \mathfrak{l}_i^{s_i},$		
$\mathsf{pk} = \mathsf{sk} * E \xrightarrow{\mathsf{pk}}$	pk	
$t_i \xleftarrow{\$} \mathbb{Z}$		
$\mathsf{esk} = \prod \mathfrak{l}_i^{t_i},$		
$\mathsf{epk}_1 = \mathsf{esk} * E, \xrightarrow{\mathsf{epk}_1}$		$c \xleftarrow{\$} \{0, 1\}$
$\mathsf{epk}_2 = \mathsf{esk} \cdot \mathsf{sk}^{-c} \xleftarrow{c}$		
$\xrightarrow{\mathsf{pk}, \mathsf{epk}_1, \mathsf{epk}_2}$		check:
		$\mathsf{epk}_1 = \mathsf{epk}_2 * ([\mathsf{sk}^c] * E).$

After k challenges c, an imposter succeeds with prob 2^{-k}.

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Hard Problem in CSIDH, CSI-FiSh, etc:
Given elliptic curves E and $E' \in S$, find $\mathfrak{a} \in H$ such that
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\begin{tikzcd}
 & E_A = \alpha * E_0 \arrow[r, "\beta * (-)"] \arrow[d, dashed, shift right] & \beta * E_A \arrow[dd, phantom, "\wr\|"] \\
E_0 \arrow[ur, "\alpha * (-)"] \arrow[dr, "\beta * (-)"'] & \boxed{E_A, E_B} \arrow[u, dashed, shift right] \arrow[d, dashed, shift right] & \\
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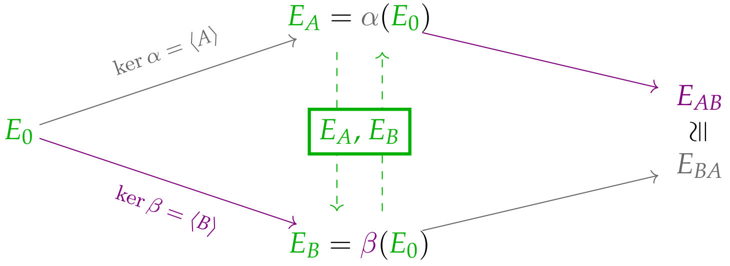

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CRS or CSIDH

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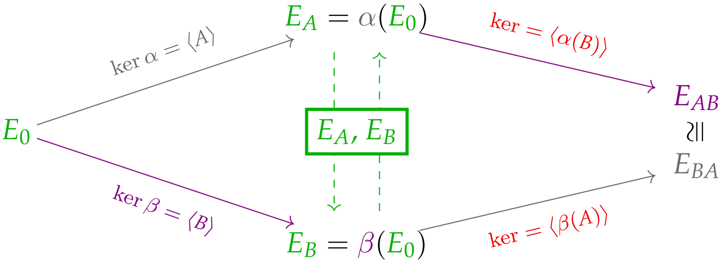

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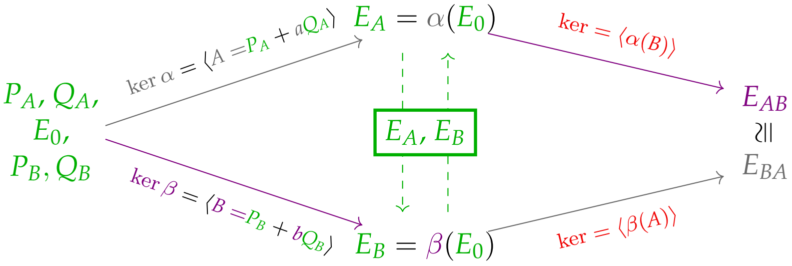
E_0 \arrow[ur, "\ker \alpha = \langle A \rangle"] \arrow[dr, "\ker \beta = \langle B \rangle"'] & \boxed{E_A, E_B} & E_{AB} \cong E_{BA} \\
 & E_B = \beta(E_0) \arrow[ur, "\ker = \langle \beta(A) \rangle"'] \arrow[uu, dashed, shift right] &
\end{tikzcd}
$$

Colour code: Public, Alice's secret, Bob's secret, ?!

Evolution of key exchange

From CRS to SIDH



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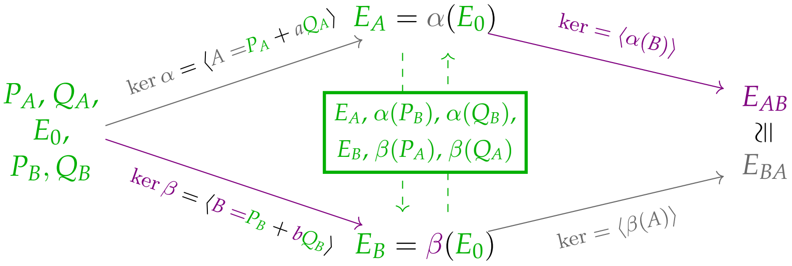

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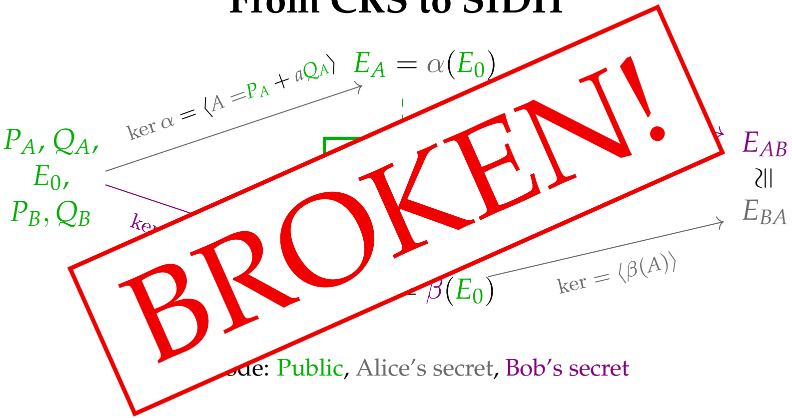

Evolution of key exchange

From CRS to SIDH

$$
\begin{tikzcd}
 & E_A = \alpha(E_0) \arrow[dr, "\ker = \langle \alpha(B) \rangle"] & \\
\begin{matrix} P_A, Q_A, \\ E_0, \\ P_B, Q_B \end{matrix} \arrow[ur, "\ker \alpha = \langle A = P_A + aQ_A \rangle"] \arrow[ddr, "\ker \beta = \langle B = P_B + bQ_B \rangle"'] & \boxed{\begin{matrix} E_A, \alpha(P_B), \alpha(Q_B), \\ E_B, \beta(P_A), \beta(Q_A) \end{matrix}} \arrow[u, dashed] \arrow[dd, dashed] & E_{AB} \arrow[d, phantom, "\wr\|"] \\
 & & E_{BA} \\
 & E_B = \beta(E_0) \arrow[ur, "\ker = \langle \beta(A) \rangle"'] &
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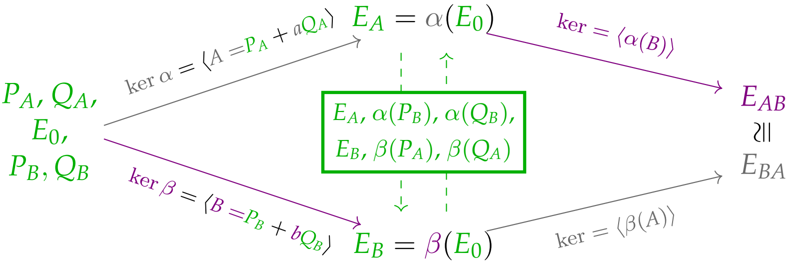
Evolution of key exchange

SIDH



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Summary of hard problems

- CRS / CSIDH – Finding α given E and $\alpha * E$.

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There are public elliptic curves E_0 and E_A, and a secret isogeny $\alpha : E_0 \to E_A$. Given the points P_B, Q_B on E_0 and $\alpha(P_B)$, $\alpha(Q_B)$, compute α. (modulo technical restrictions)*

*Details for the elliptic curve lovers:

p a large prime; $E_0/\mathbb{F}_{p^2}$ and $E_A/\mathbb{F}_{p^2}$ supersingular; $\deg(\alpha)$, B public large smooth coprime integers; points P_B, Q_B chosen such that $\langle P_B, Q_B \rangle = E_0[B]$.

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History of the SIDH problem

2011 Problem introduced by De Feo, Jao, and Plut

2016 Galbraith, Petit, Shani, Ti give active attack

2017 Petit gives passive attack on some parameter sets

2020 de Quehen, Kutas, Leonardi, M., Panny, Petit, Stange give passive attack on more parameter sets

2022 Castryck-Decru and Maino-M. give passive attack on SIKE parameter sets; Robert extends to all parameter sets

- CD and MM attack is subexponential in most cases
- CD attack polynomial-time when $\mathrm{End}(E_0)$ known
- Robert attack polynomial-time in all cases
- Panny and Pope implement MM attack; Wesolowski independently discovers direct recovery method

Petit's trick: torsion points to isogenies

Finding the secret isogeny α of known degree, given $\alpha(E_0[B])$.

$$\begin{tikzcd} E_0 \arrow[rr, bend left, "\alpha"] & & E_A \arrow[ll, bend left, "\widehat{\alpha}"] \end{tikzcd}$$

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- Restriction # 1: Assume we can choose $\iota : E_0 \to E_0$.

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- Know $\alpha(E_0[B])$ (and $\widehat{\alpha}(E_A[B])$) from public torsion points.

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- Restriction # 2: If there exist ι, n such that $\deg(\theta) = B$, then can completely determine θ, and α, in polynomial-time.
- Restriction # 2 rules out SIKE parameters, where $B \approx \deg(\alpha)$ (and $p \approx B \cdot \deg \alpha$).

Enter Kani

There are public elliptic curves E_0 and E_A, and a secret isogeny $\alpha : E_0 \to E_A$. Given the points P_B, Q_B on E_0 and $\alpha(P_B)$, $\alpha(Q_B)$, compute α. (modulo technical restrictions)*

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Problem:
Not enough choices $\theta : E_A \to E_A$.
'No θ of degree N.'

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Solution? $\theta : E_0 \times E_A \to E_0 \times E_A$?
$\rightsquigarrow$ still not enough.

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Not enough choices $\theta : E_A \to E_A$.
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Solution? $\theta : E_0 \times E_A \to E_0 \times E_A$?
$\rightsquigarrow$ still not enough. But! Kani's lemma:

- Constructs E_1, E_2 such that there exists a (structure-preserving) isogeny

$$E_1 \times E_A \to E_0 \times E_2$$

  of the right degree, N^2.
- Petit's trick then applies.

Recovering the secret

Finding the secret isogeny α of known degree.

$$\begin{tikzcd}
E_0 \arrow[loop left, "\iota"] \arrow[rr, bend left, "\alpha"] & & E_A \arrow[ll, bend left, "\widehat{\alpha}"]
\end{tikzcd}$$

Recovering the secret

Finding the secret isogeny α of known degree.

$$\begin{tikzcd} E_0 \arrow[rr, bend left, "\alpha"] & & E_A \\ E_1 \arrow[u, "\varphi"] & & \end{tikzcd}$$

Recovering the secret

Finding the secret isogeny α of known degree.

$$\begin{tikzcd}
E_0 \arrow[r, bend left, "\alpha"] & E_A \\
E_1 \arrow[u, dashed, "\varphi"] \arrow[ur, dashed, "f"] \arrow[r, dashed] & E_2 \arrow[u, dashed]
\end{tikzcd}$$

Kani's lemmaconstructs the above such that

$$\Phi = \begin{pmatrix} \varphi & -\widehat{\alpha} \\ * & * \end{pmatrix} : E_1 \times E_A \to E_0 \times E_2$$

is a structure preserving isogeny of degree N^2, and

$$\ker(\Phi) = \{(\deg(\alpha)P, f(P)) : P \in E_1[N]\}$$

$\rightsquigarrow$ can compute Φ and read off secret α!

Recovering the secret with Robert's trick

Finding the secret isogeny α of known degree.

$$\begin{tikzcd}
{E_0}^4 \arrow[rrr, bend left, "\alpha^4"] & & & {E_A}^4 \\
E_0^4 \arrow[u, dashed, "\varphi"] \arrow[urrr, dashed, "f"] \arrow[rrr, dashed] & & & E_A^4 \arrow[u, dashed]
\end{tikzcd}$$

constructs the above such that

$$\Phi = \begin{pmatrix} \varphi & -\widehat{\alpha}^4 \\ * & * \end{pmatrix} : E_0^4 \times E_A^4 \to E_0^4 \times E_A^4$$

is a structure preserving isogeny of degree N^2, and

$$\ker(\Phi) \text{ is known}$$

$\rightsquigarrow$ can compute Φ and read off secret α!

Power unleashed

Consequence 1: Factoring isogenies.

$$\begin{tikzcd}
E_0 \arrow[rrr, "\alpha"] & & & E_A \\
E_1 \arrow[u, "\varphi"] \arrow[urrr, dashed, "f"] \arrow[rrr, dashed] & & & E_2 \arrow[u, dashed]
\end{tikzcd}$$

Kani's lemma states that

$$\Phi = \begin{pmatrix} \varphi & -\widehat{\alpha} \\ * & * \end{pmatrix} : E_1 \times E_A \to E_0 \times E_2$$

is a structure preserving isogeny of degree B^2, and

$$\ker(\Phi) = \{(\deg(\alpha)P, f(P)) : P \in E_1[B]\}.$$

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Kani's lemma states that

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is a structure preserving isogeny of degree B^2, and

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Power unleashed

Consequence 2: Let

- $\alpha : E_0 \to E_A$ be an isogeny.
- B a smooth integer, $\langle P_B, Q_B \rangle = E_0[B]$.

Then:

- α can be stored efficiently as $\alpha(P_B)$, $\alpha(Q_B)$.
- images under α can be efficiently computed from this representation.
  Doesn't require $\deg(\alpha)$ to be smooth!

QFESTA: a PKE

Colour code: Public, Alice's secret, Bob's secret, unknown

Alice: **KeyGen**

E_0

,

,

QFESTA: a PKE

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Alice: **KeyGen**

$$E_0 \xrightarrow{\varphi_{A,d_{A,1}}} E_{A,1}$$

,

,

QFESTA: a PKE

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Alice: **KeyGen**

$$E_0 \xrightarrow{\varphi_{A,d_{A,1}}} E_{A,1} \xrightarrow{\varphi_{A,3^b}} E_A$$

,

,

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Alice: **KeyGen**

$$E_0 \xrightarrow{\varphi_{A,d_{A,1}}} E_{A,1} \xrightarrow{\varphi_{A,3^b}} E_A \quad \begin{pmatrix} \mu_1 & \\ & \mu_2 \end{pmatrix}^* \quad E_A$$

,

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$$E_0 \xrightarrow{\varphi_{A,d_{A,1}}} E_{A,1} \xrightarrow{\varphi_{A,3^b}} E_A \xrightarrow{\begin{pmatrix} \mu_1 & \\ & \mu_2 \end{pmatrix}_*} E_A$$

$$P_0, Q_0 \qquad\qquad P_{A,1}, Q_{A,1} \qquad\qquad P_A, Q_A \qquad\qquad \mu_1 P_A, \mu_2 Q_A$$

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P_0, Q_0 $\qquad$ $P_{A,1}, Q_{A,1}$ $\qquad$ P_A, Q_A $\qquad$ $\mu_1 P_A, \mu_2 Q_A$

- $\mathsf{sk}_A \leftarrow E_{A,1}, P_{A,1}, Q_{A,1}, \begin{pmatrix} \mu_1 & \\ & \mu_2 \end{pmatrix},$

  ,

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,

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Bob: **Encrypt** $B \in \mathbf{Mat}_{2\times 2}[\mathbb{Z}/2^{3a}\mathbb{Z}]$

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- $\mathsf{sk}_A \leftarrow E_{A,1}, P_{A,1}, Q_{A,1}, \begin{pmatrix} \mu_1 & \\ & \mu_2 \end{pmatrix}$, $\mathsf{pk}_A \leftarrow E_A, \mu_1 P_A, \mu_2 Q_A$
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$$
\begin{tikzcd}
E_0 \arrow[r, "\varphi_{A,d_{A,1}}"] \arrow[d, dashed, "\varphi_{B,d_1}"'] & E_{A,1} \arrow[r, "\varphi_{A,3^b}"] & E_A & \begin{pmatrix} \mu_1 & \\ & \mu_2 \end{pmatrix}^* & E_A \arrow[d, dashed, "\varphi_{B,3^{2b}}"] \\
E_1 & & & & E_2
\end{tikzcd}
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E_1 \arrow[d, dashed, "B*"'] & & & E_2 \arrow[d, dashed, "B*"] \\
E_1 & & & E_2
\end{tikzcd}$$

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\begin{matrix} P_1 \\ Q_1 \end{matrix} & E_1 \arrow[d, dashed, "B*"] & & & E_2 \arrow[d, dashed, "B*"] & \begin{matrix} P_2 \\ Q_2 \end{matrix} \\
\begin{matrix} P_{1,B} \\ Q_{1,B} \end{matrix} & E_1 & & & E_2 & \begin{matrix} P_{2,B} \\ Q_{2,B} \end{matrix}
\end{tikzcd}$$

- $\mathsf{sk}_A \leftarrow E_{A,1}, P_{A,1}, Q_{A,1}, \begin{pmatrix} \mu_1 & \\ & \mu_2 \end{pmatrix}$, $\mathsf{pk}_A \leftarrow E_A, \mu_1 P_A, \mu_2 Q_A$
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\begin{smallmatrix} P_1 \\ Q_1 \end{smallmatrix} & E_1 \arrow[d, "B*"] & & & E_2 \arrow[d, "B*"] & \begin{smallmatrix} P_2 \\ Q_2 \end{smallmatrix} \\
\begin{smallmatrix} P_{1,B} \\ Q_{1,B} \end{smallmatrix} & E_1 & & & E_2 & \begin{smallmatrix} P_{2,B} \\ Q_{2,B} \end{smallmatrix}
\end{tikzcd}
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- $B \in \mathrm{Mat}_{2\times 2}$, $\mathrm{enc}(B) \leftarrow E_1, P_{1,B}, Q_{1,B}, E_2, P_{2,B}, Q_{2,B}$

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$$
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E_0 \arrow[r, "\varphi_{A,d_{A,1}}"] \arrow[d, dashed, "\varphi_{B,d_1}"] & E_{A,1} \arrow[r, "\varphi_{A,3^b}"] & E_A \cdot \begin{pmatrix} \mu_1 & \\ & \mu_2 \end{pmatrix} * \cdot E_A \arrow[d, dashed, "\varphi_{B,3^{2b}}"] \\
E_1 \arrow[d, dashed, "B*"] & & E_2 \arrow[d, dashed, "B*"] \\
E_1 & & E_2
\end{tikzcd}
$$

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$$
\begin{tikzcd}
E_0 \arrow[r, "\varphi_{A,d_{A,1}}"] \arrow[d, dashed, "\varphi_{B,d_1}"'] & E_{A,1} \arrow[r, "\varphi_{A,3^b}"] \arrow[d, dashed, "B*"] & E_A \arrow[r, dotted, "\begin{pmatrix}\mu_1 & \\ & \mu_2\end{pmatrix}*"] & E_A \arrow[d, dashed, "\varphi_{B,3^{2b}}"] \\
E_1 \arrow[d, dashed, "B*"'] & E_{A,1} \arrow[rrd, "\deg = 3^{3b}"] & & E_2 \arrow[d, dashed, "B*"] \\
E_1 \arrow[ru, "\deg = d_1 d_{A,1}"] & & & E_2
\end{tikzcd}
$$

$P_{1,B}, Q_{1,B}$ $\qquad\qquad$ $\mu_1^{-1} d_1 P_{2,B}, \mu_2^{-1} d_1 Q_{2,B}$

- $\mathsf{sk}_A \leftarrow E_{A,1}, P_{A,1}, Q_{A,1}, \begin{pmatrix}\mu_1 & \\ & \mu_2\end{pmatrix}$, $\mathsf{pk}_A \leftarrow E_A, \mu_1 P_A, \mu_2 Q_A$
- $B \in \mathrm{Mat}_{2\times 2}$, $\mathrm{enc}(B) \leftarrow E_1, P_{1,B}, Q_{1,B}, E_2, P_{2,B}, Q_{2,B}$

QFESTA: a PKE

Colour code: Public, Alice's secret, Bob's secret, unknown

Alice: **Decrypt** B

$$
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\begin{tikzcd}
 & P_{A,1}, Q_{A,1} & & \\
E_0 \arrow[r, "\varphi_{A,d_{A,1}}"] \arrow[d, "\varphi_{B,d_1}"] & E_{A,1} \arrow[r, "\varphi_{A,3^b}"] \arrow[d, "B*"] & E_A \cdot \begin{pmatrix} \mu_1 & \\ & \mu_2 \end{pmatrix} * \cdot & E_A \arrow[d, "\varphi_{B,3^{2b}}"] \\
E_1 \arrow[d, "B*"] & E_{A,1} \arrow[rrd, "\deg = 3^{3b}"] & & E_2 \arrow[d, "B*"] \\
P_{1,B},\ Q_{1,B} \quad E_1 \arrow[ru, "\deg = d_1 d_{A,1}"] & & & E_2
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Summary

Three main tools in isogeny-based cryptography:

- The class-group action.
  - NIKE: CRS, CSIDH, CSURF, SQALE, OSIDH (cf. Eli)
  - Signatures: Seasign, CSI-FISh, SCALLOP
- The Deuring correspondence.
  - Signatures: SQISign, SQISign2D (also uses Kani)
- Kani's lemma.
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Thank you!

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