Lattice Cryptography (1. Public Key Encryption)

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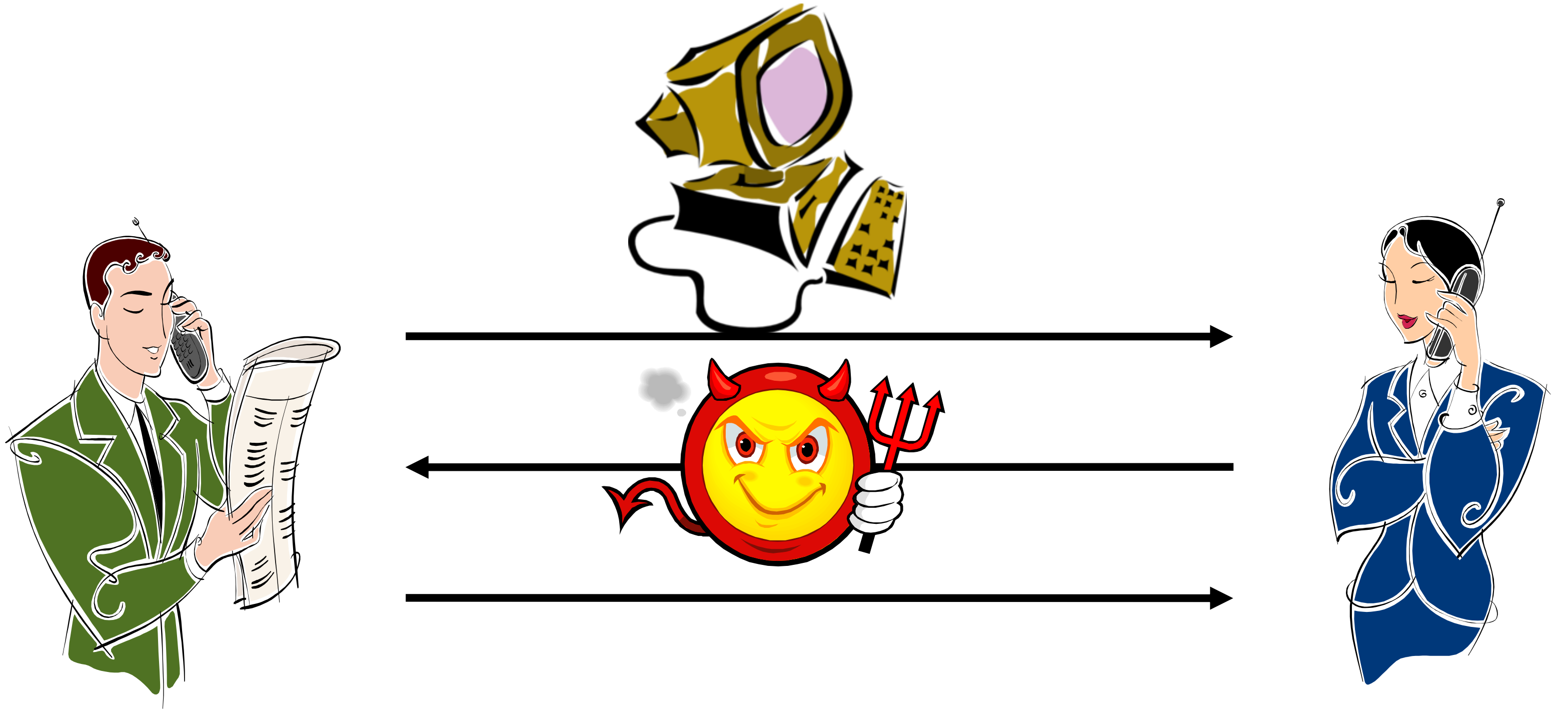
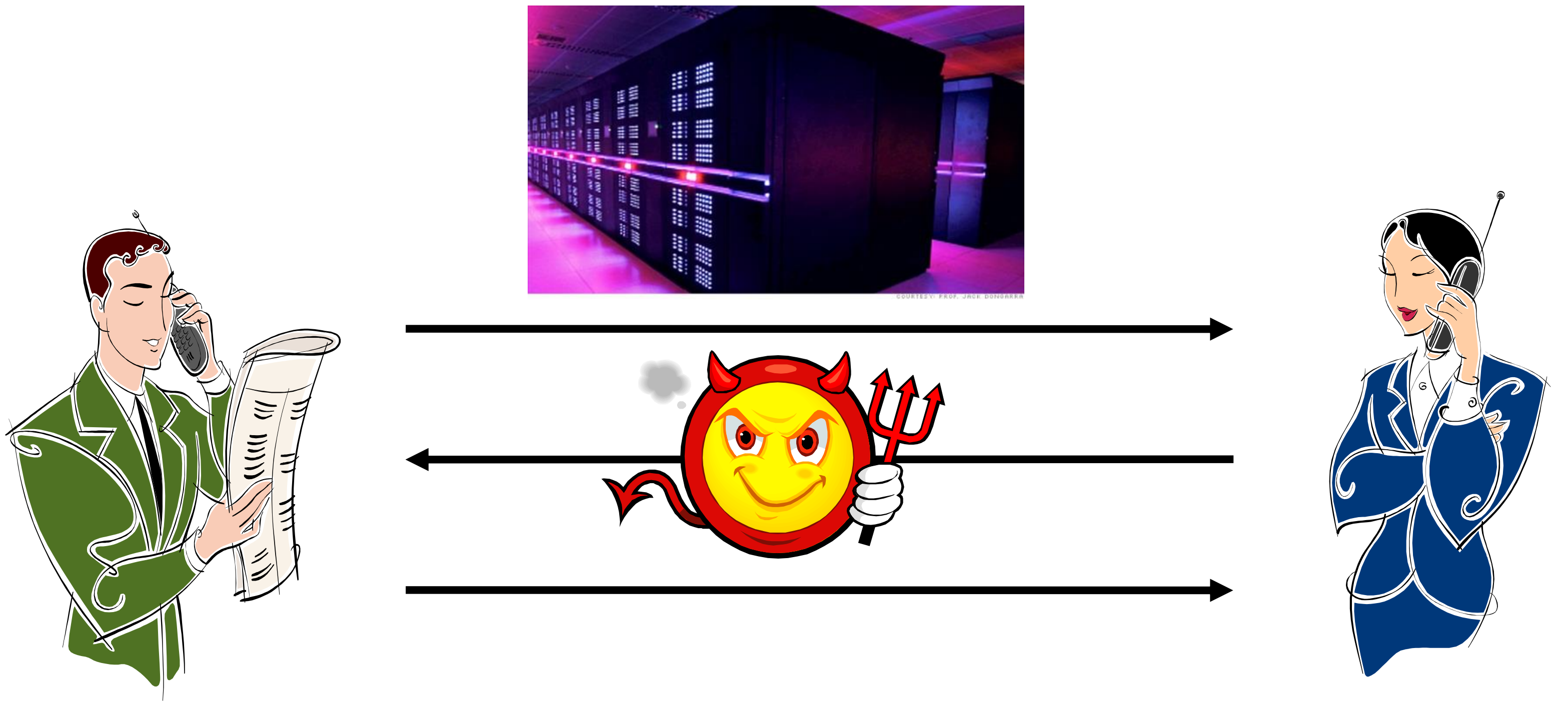
Cryptography

Allows for secure communication in the presence of malicious parties



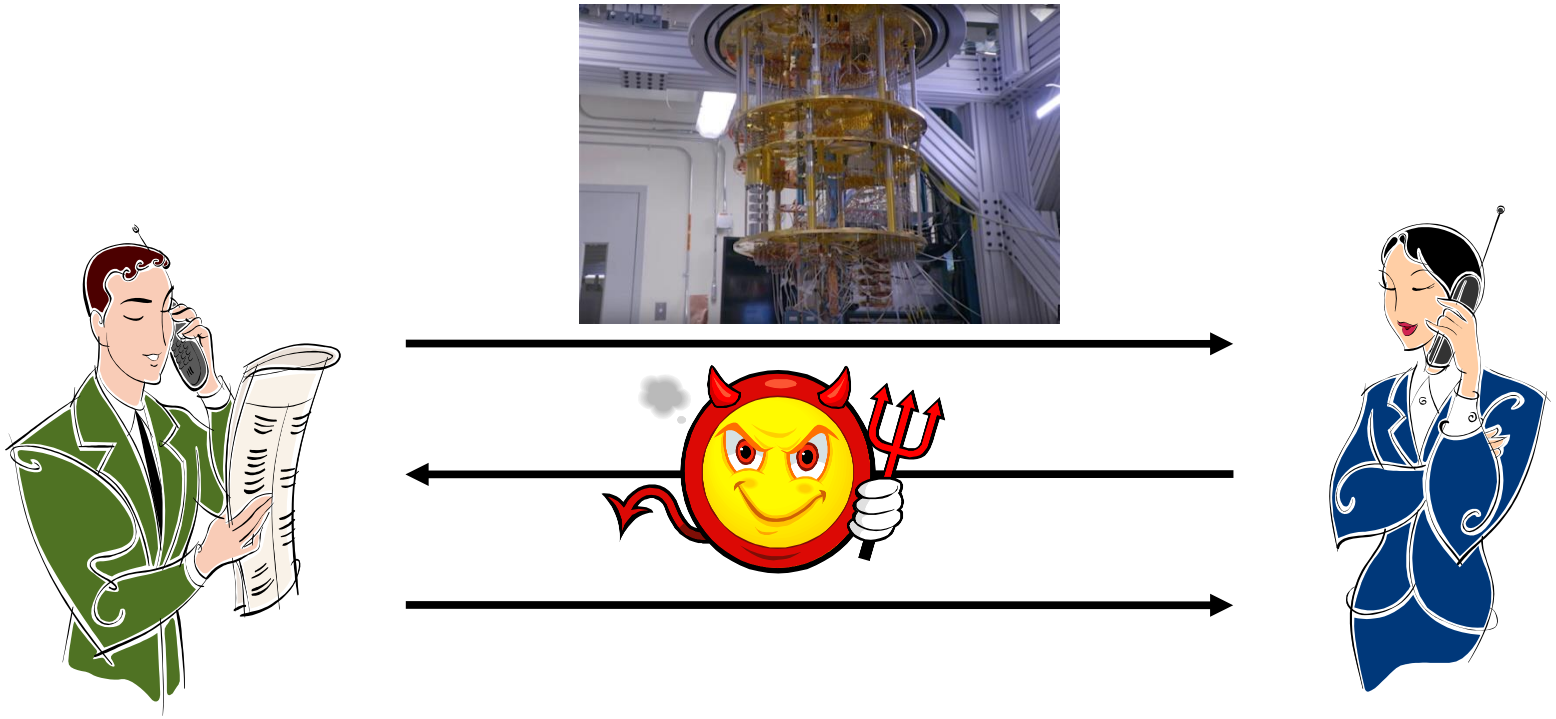
Cryptography

Large increase in the adversary's computing power requires only a small increase in the key size



Cryptography

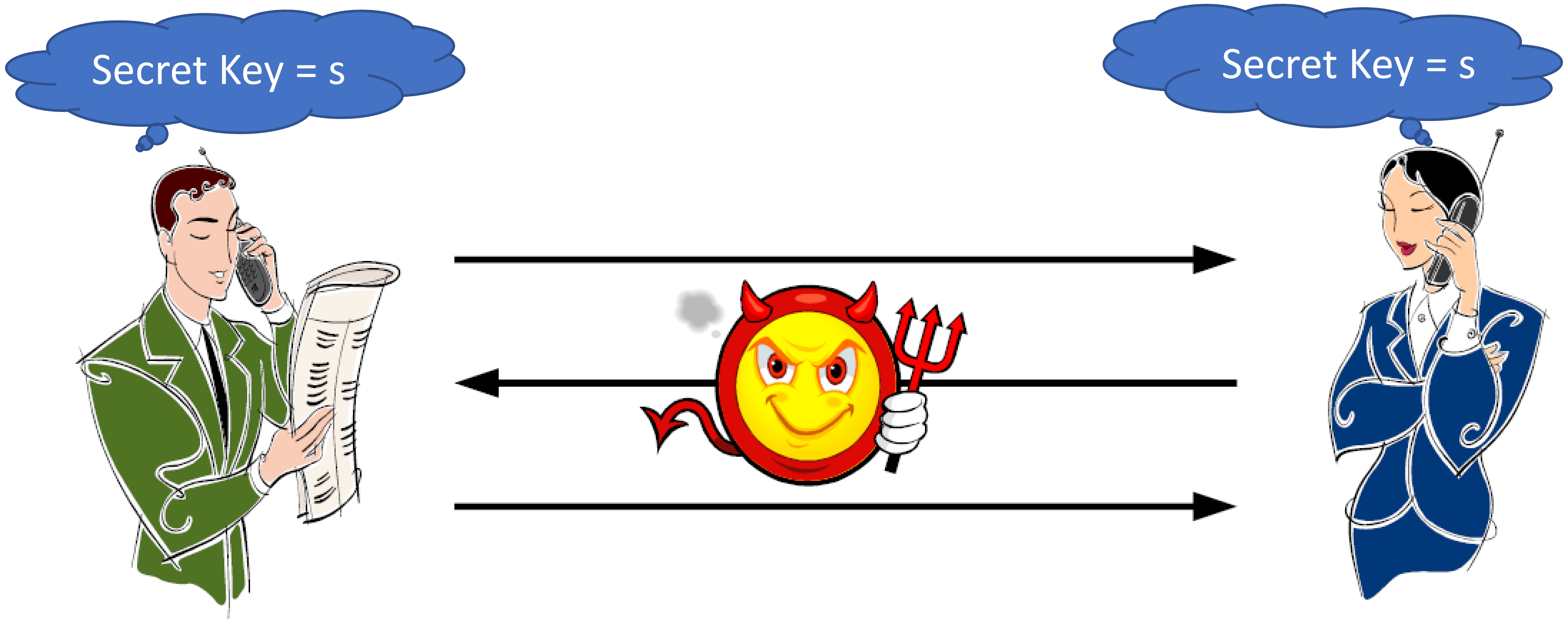
A quantum computer is outside the standard model of computation for efficiency purposes



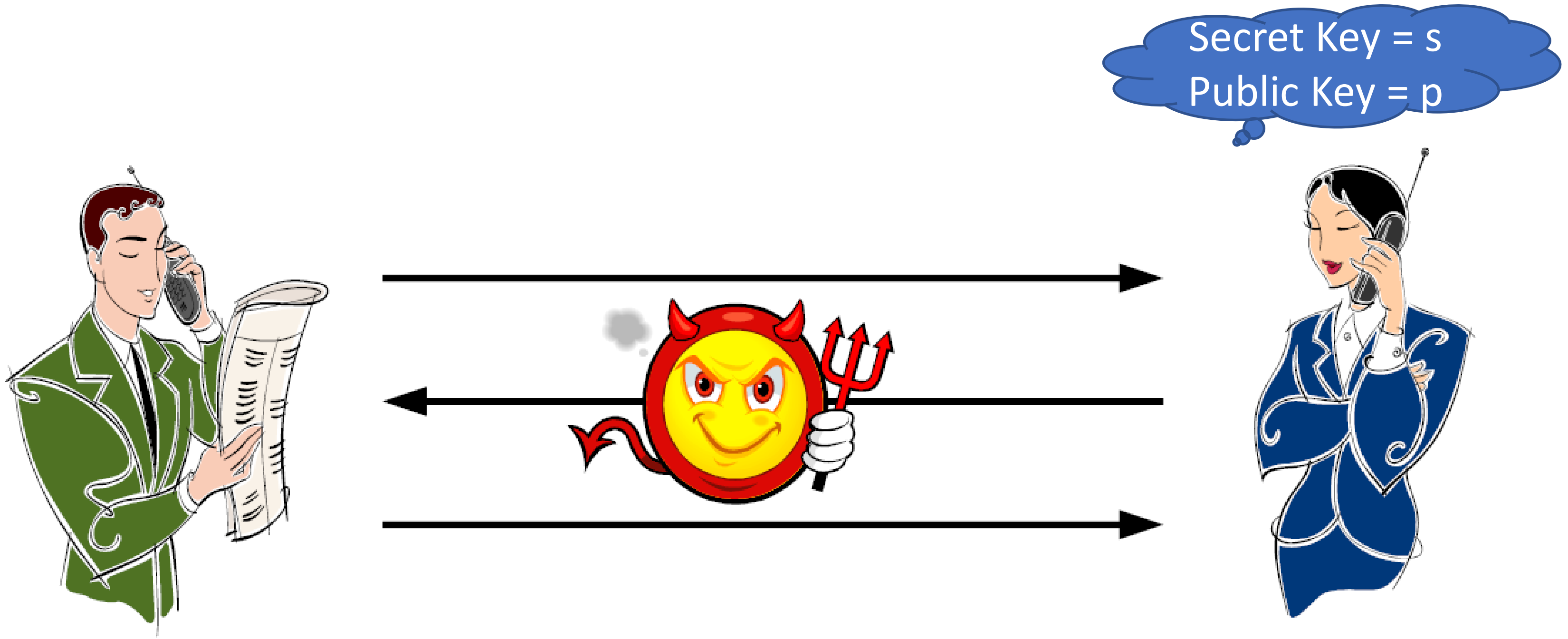
Symmetric-Key Cryptography

Symmetric-Key Cryptography

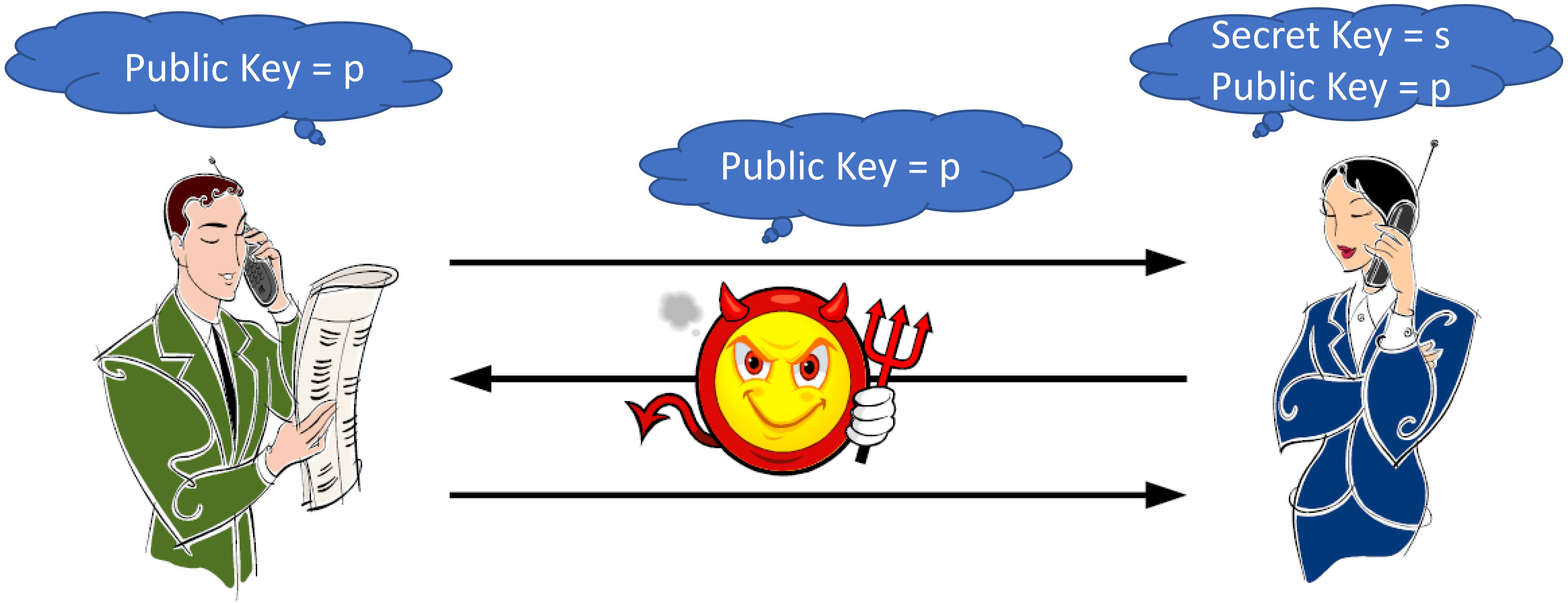
Will still exist if quantum computers are built



Public-Key Cryptography



Public-Key Cryptography



Mathematical Assumptions for Public-Key Cryptography

~~Factoring is hard~~

~~$N=pq$~~

~~Computing discrete logs is hard~~

~~$g^x=y \bmod p$~~

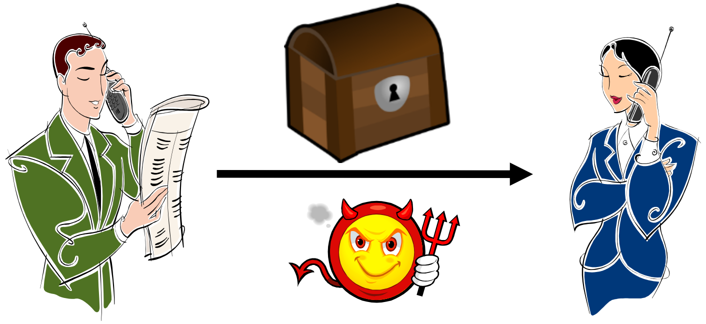
Mostly problems from number theory

All broken once a quantum computer is built

Consequence of quantum computing

Current public key schemes will be broken

Quantum computers will recover all of **today’s** secrets



Do not need quantum to defend against quantum

Quantum computers are not all-powerful.

They simply solve some problems faster.

Base cryptography on problems they don't solve.

How do we know that (quantum) computers don't solve a problem?

We don't … all we can say is that researchers tried to solve the problem for X decades and failed.

Effect of quantum computers

- Symmetric Cryptography
  - Mostly fine
- Public-Key Cryptography
  - Everything used today – broken!

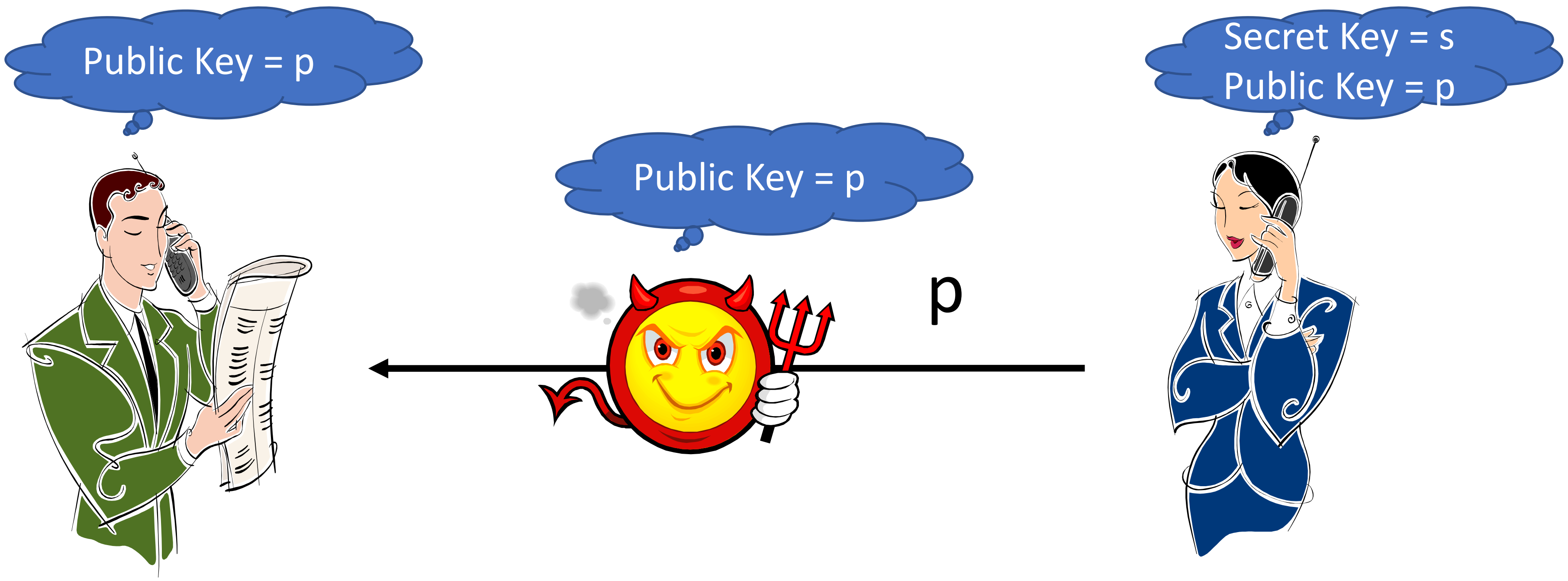
Timeline for change to Quantum-Safe (post-quantum) Crypto:

Algorithm Selection: 2017 – 2022

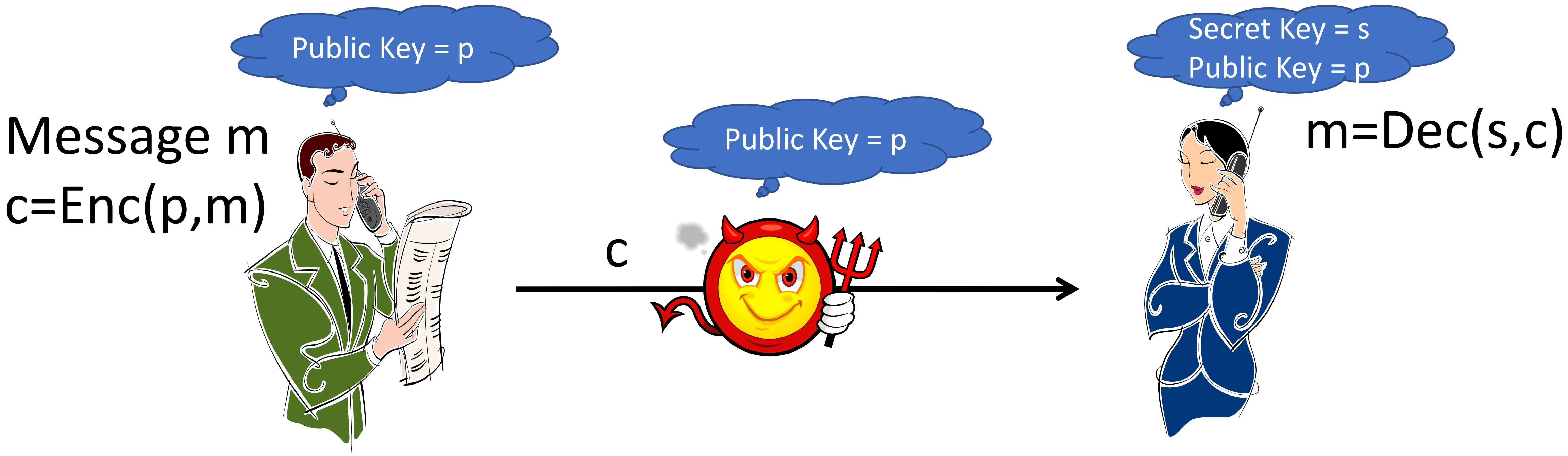
Writing Standards: 2022 – 2024

Transition should be complete (NSA): 2030-2035

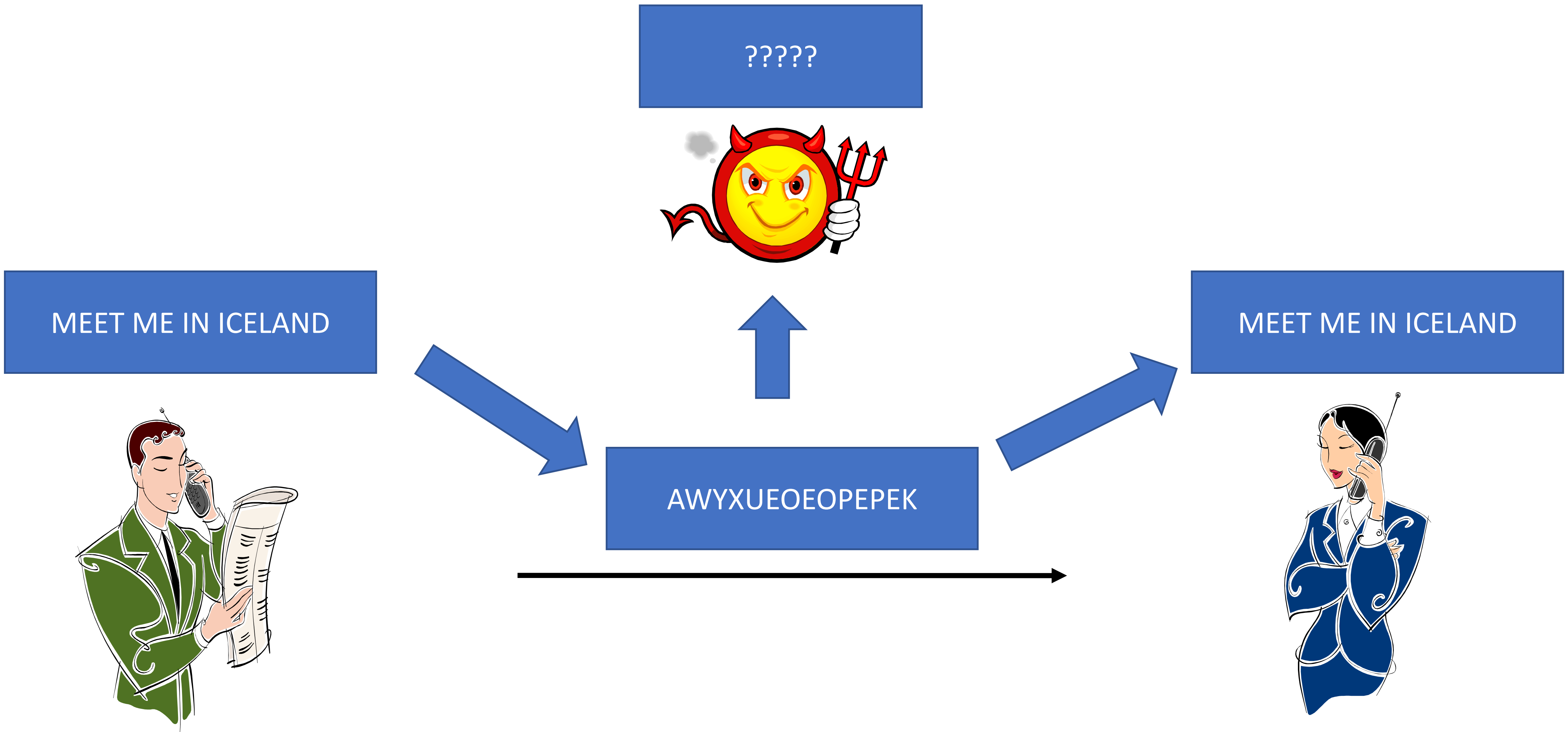
Public-Key Encryption



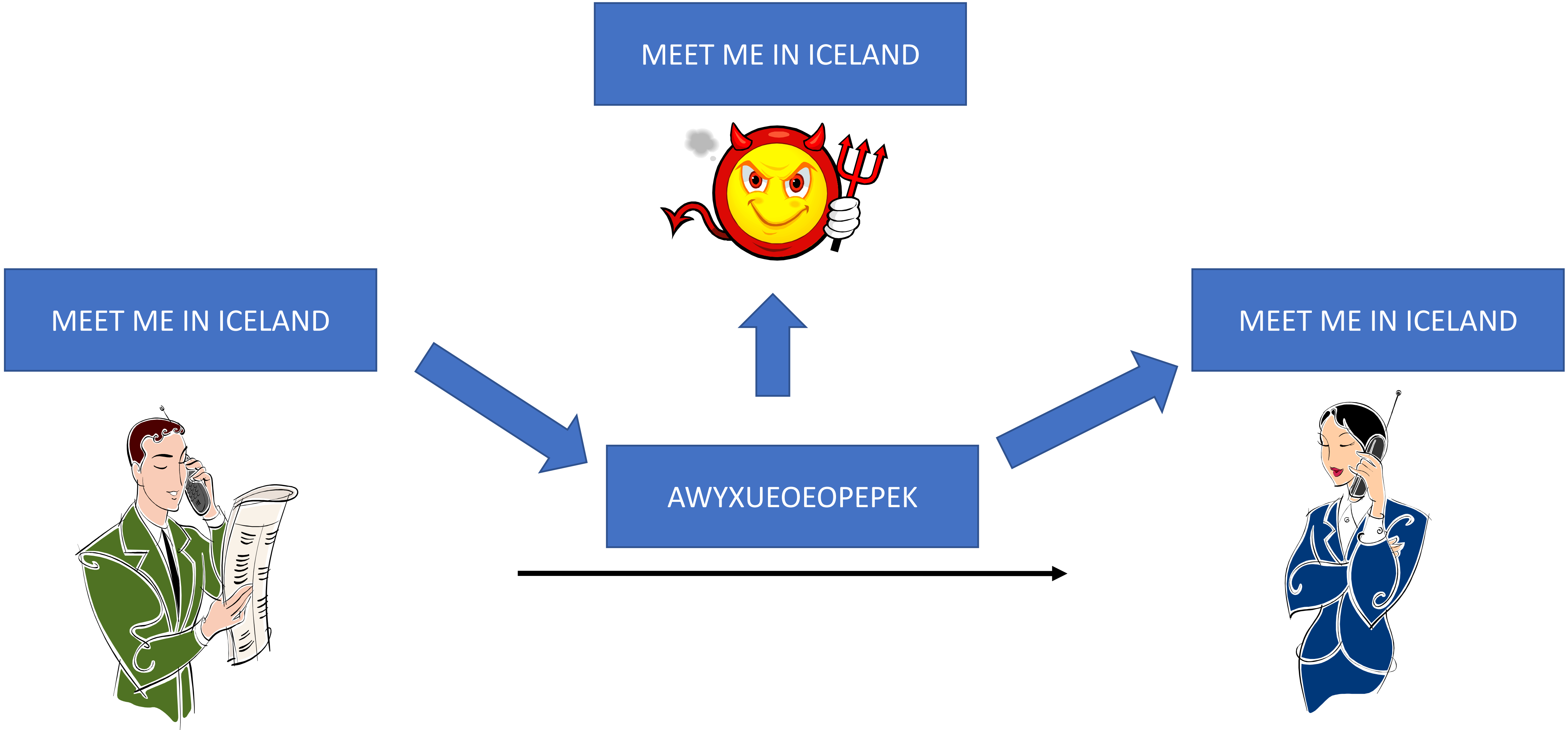
Public-Key Encryption



What is Secure Encryption?



Public Key Cryptography



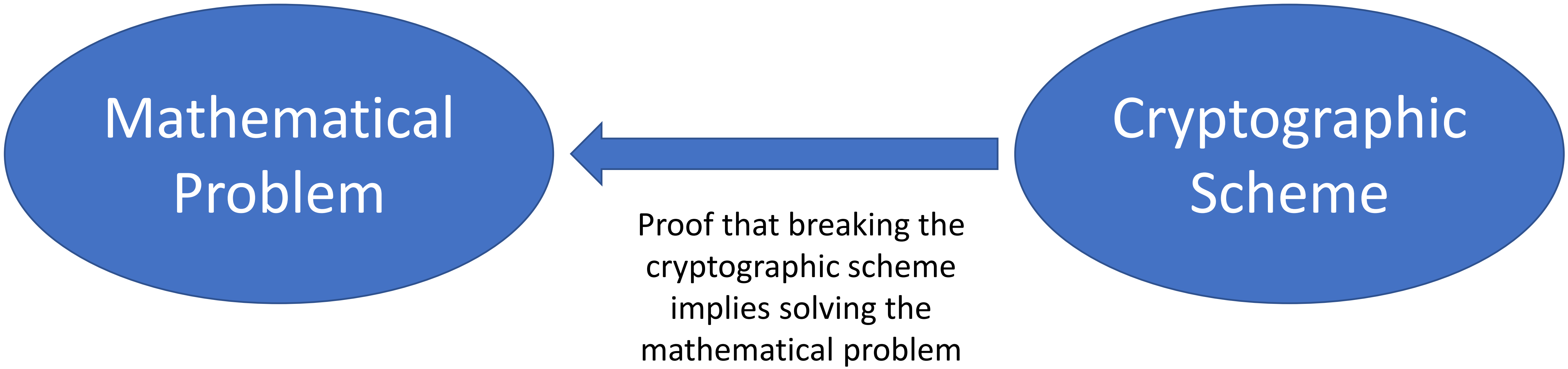
Formal Definition

For any two messages m_1 and m_2 of the adversary's choosing, he cannot **distinguish** between

- $c_1 = Enc(m_1)$
- $c_2 = Enc(m_2)$

Encryption needs to be randomized

Building Cryptography



Lattice Cryptography

- NIST had a competition to create new quantum-safe cryptographic standards
- RSA / Discrete Log / Elliptic Curve cryptography will be phased out by 2033 (at least in the US government)
- Main 2 standards - encryption and digital signature - selected are lattice-based

  (CRYSTALS-Kyber, CRYSTALS-Dilithium)

Learning with Errors Problem

Hard Problem Intuition

$$\begin{pmatrix} & & \\ & \mathbf{A} & \\ & & \end{pmatrix} \begin{bmatrix} \\ \mathbf{y} \\ \\ \end{bmatrix} = \begin{bmatrix} \mathbf{z} \\ \\ \\ \end{bmatrix} \mod p$$

Given (**A**,**z**), find **y**

Easy! Just invert **A** and multiply by **z**

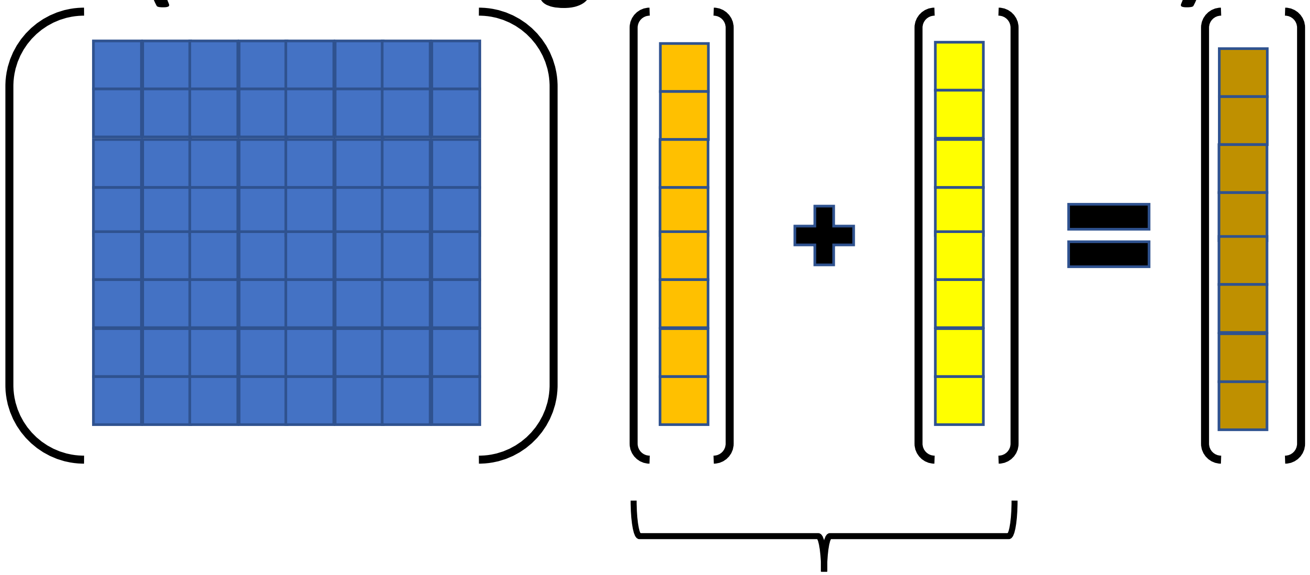
Hard Problem Intuition

$$\begin{pmatrix} & & \\ & \mathbf{A} & \\ & & \end{pmatrix} \underbrace{\begin{bmatrix} \\ \mathbf{y} \\ \\ \end{bmatrix} + \begin{bmatrix} \\ \mathbf{e} \\ \\ \end{bmatrix}}_{\text{Small coefficients}} = \begin{bmatrix} \\ \mathbf{z} \\ \\ \end{bmatrix} \mod p$$

Given (**A**,**z**), find (**y**,**e**)

Seems hard.

Hard Problem Intuition
(**Learning With Errors**)



mod p

Small coefficients to enforce uniqueness

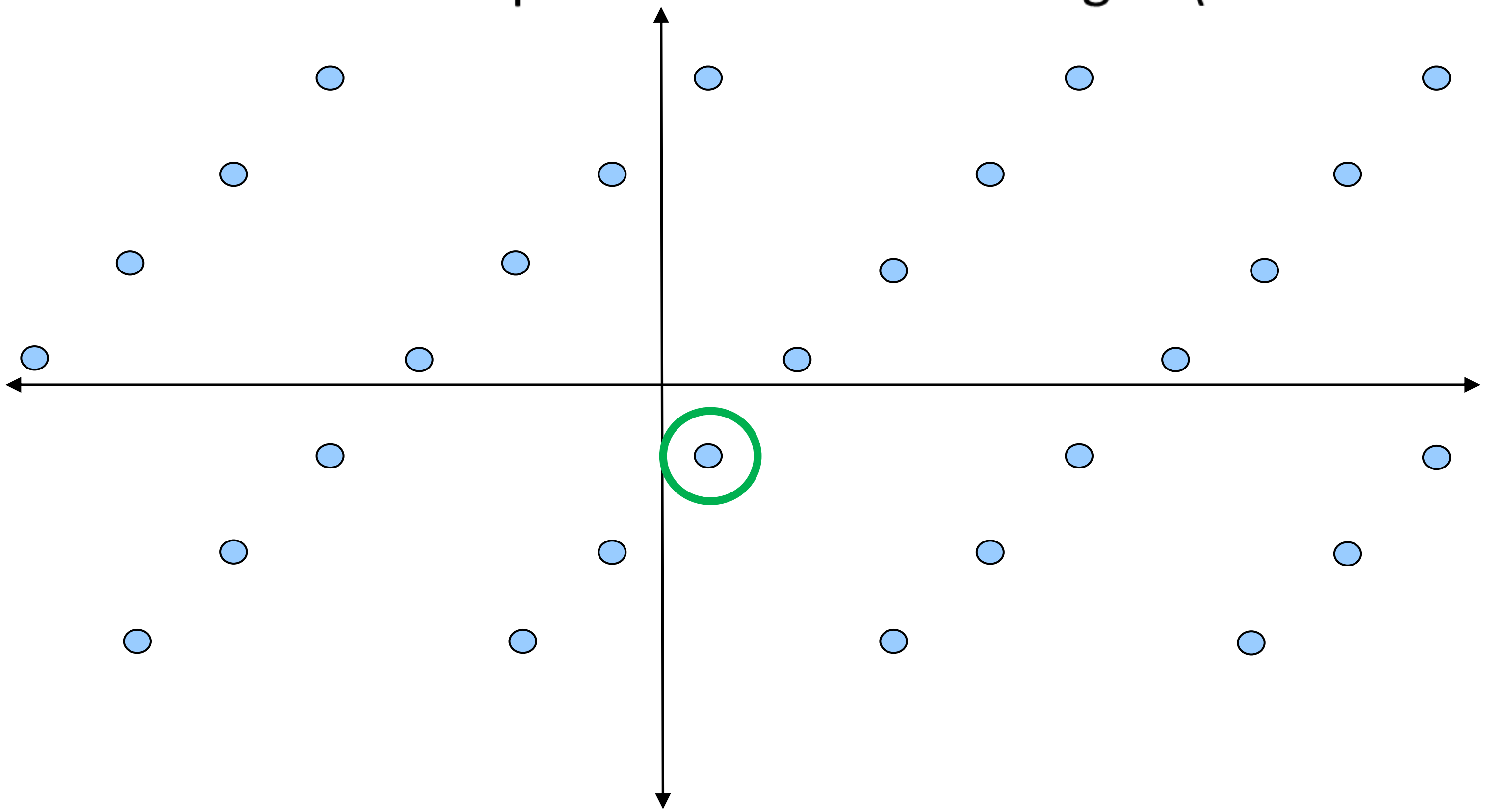
Given (**A**,**z**), find (**y**,**e**)

Seems hard.

Why is this "Lattice" Crypto?

All solutions $\binom{y}{e}$ to **Ay**+**e**=**z** mod p form a "shifted" lattice.

We want to find the point closest to the origin (BDD Problem).



Distinguishing from Random is also Hard

Search LWE Problem: Given (A,t=As+e mod p), find s

Decision LWE Problem: Given either (A, t=As+e mod p) or (A,u), where u is random mod p, figure out which tuple you have

$$\begin{pmatrix} \;\; A \;\; \end{pmatrix} \begin{bmatrix} y \end{bmatrix} + \begin{bmatrix} e \end{bmatrix} = \begin{bmatrix} z \end{bmatrix} \mod p$$

Looks Random mod p

Public Key Encryption from LWE

“Column” LWE is Pseudorandom

$$\begin{bmatrix} 4 & 6 & 7 & 9 \\ 3 & 9 & 0 & 7 \\ 8 & 4 & 6 & 5 \\ 1 & 6 & 4 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 9 \\ 8 \\ 0 \end{bmatrix}$$

“Row” LWE is also Pseudorandom

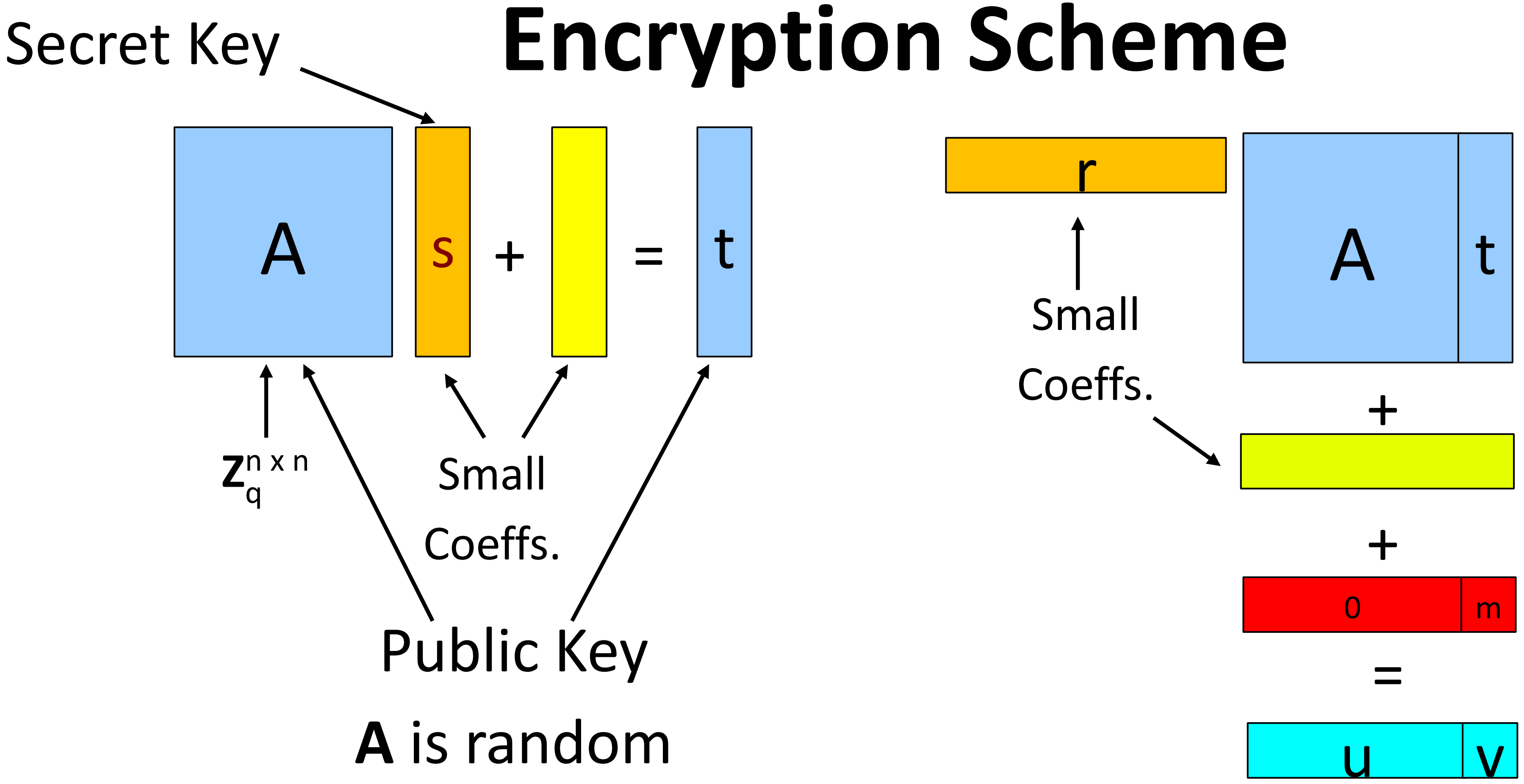
$$\begin{bmatrix}1&0&0&1\end{bmatrix}\begin{bmatrix}4&6&7&9&0\\3&9&0&7&9\\8&4&6&5&8\\1&6&4&3&0\end{bmatrix}$$

$$+\begin{bmatrix}1&1&1&0&0\end{bmatrix}$$

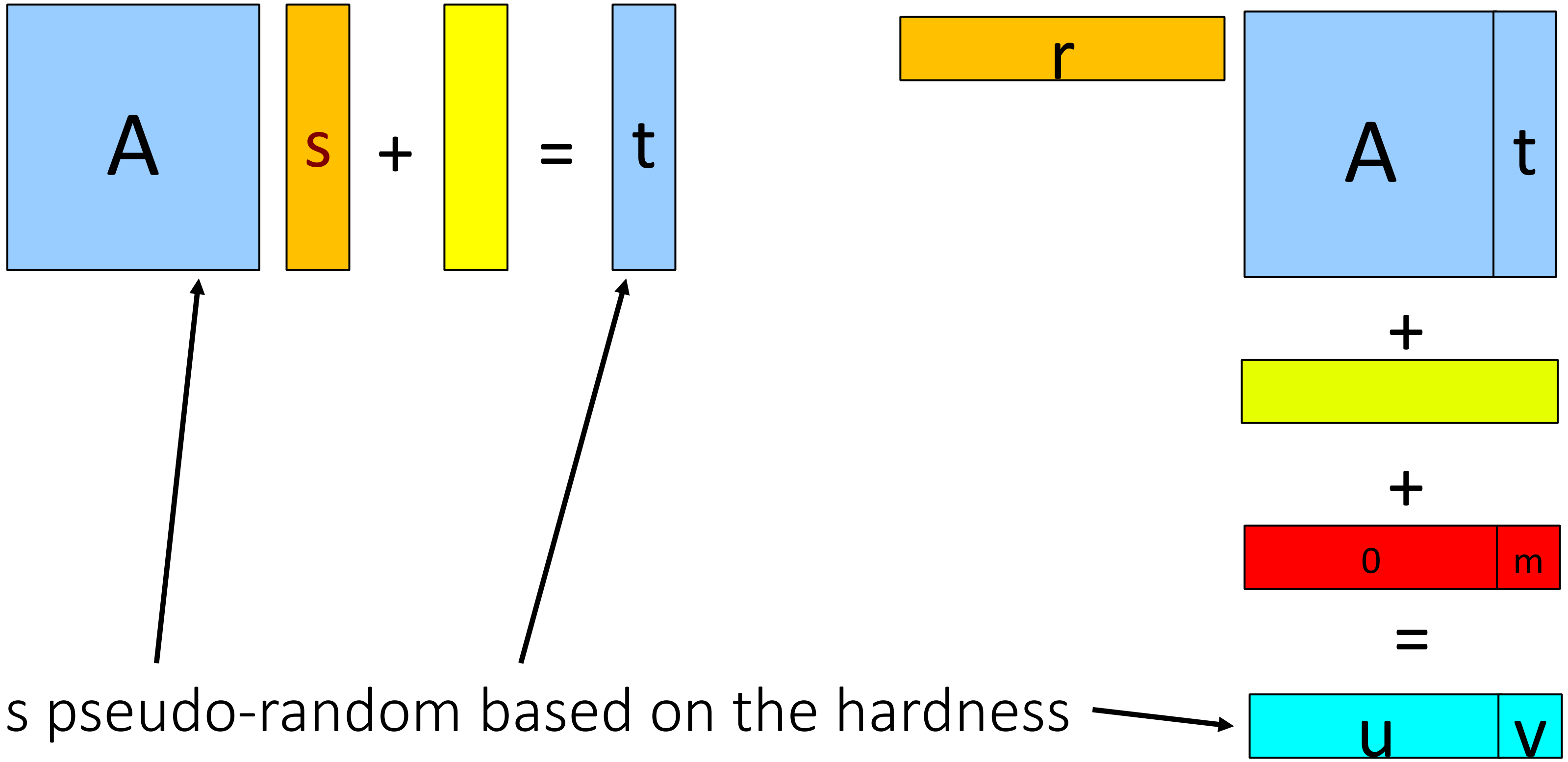
$$=\begin{bmatrix}6&3&2&2&0\end{bmatrix}$$

pseudorandom

Encryption Scheme

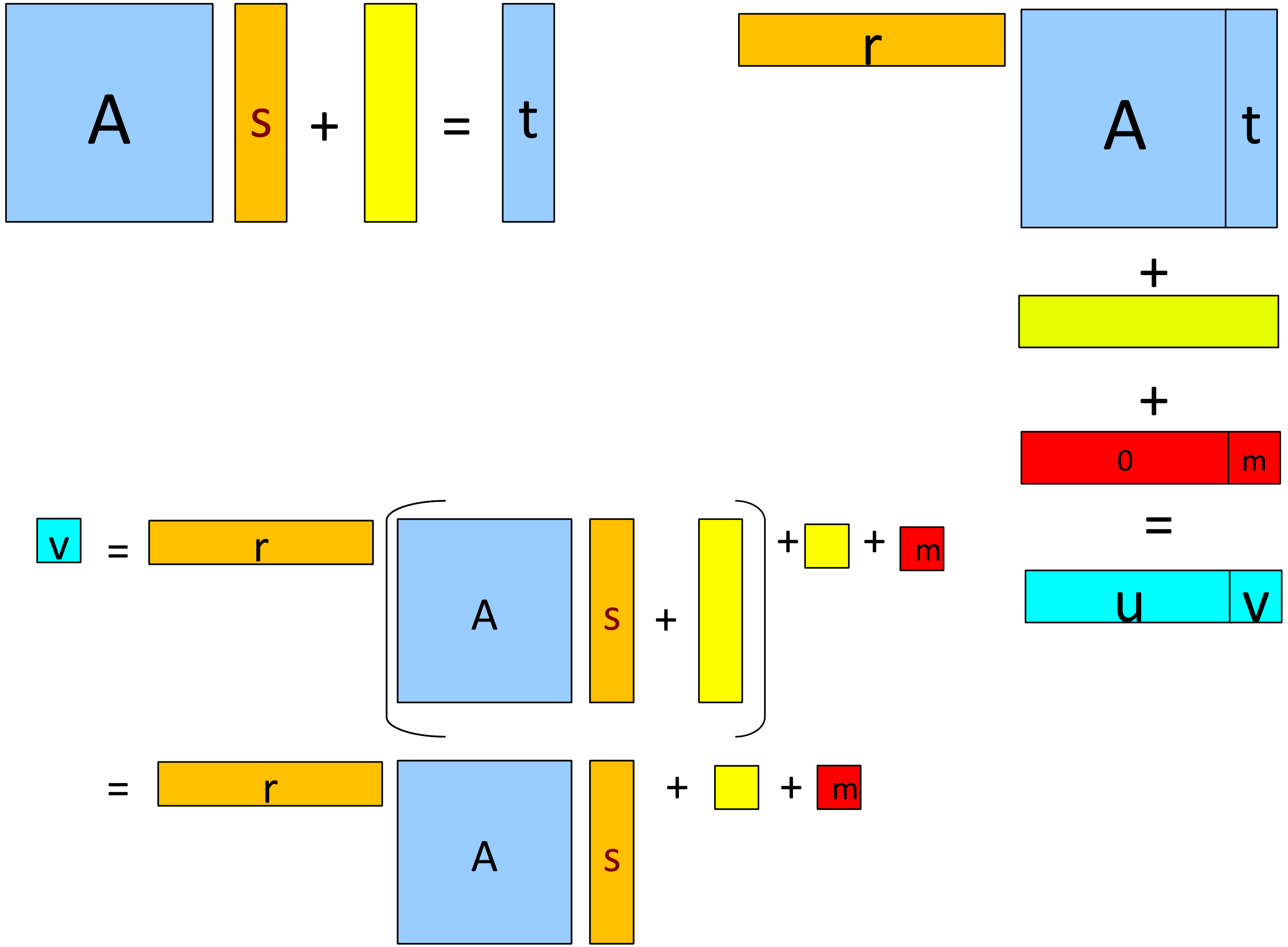


Encryption Scheme

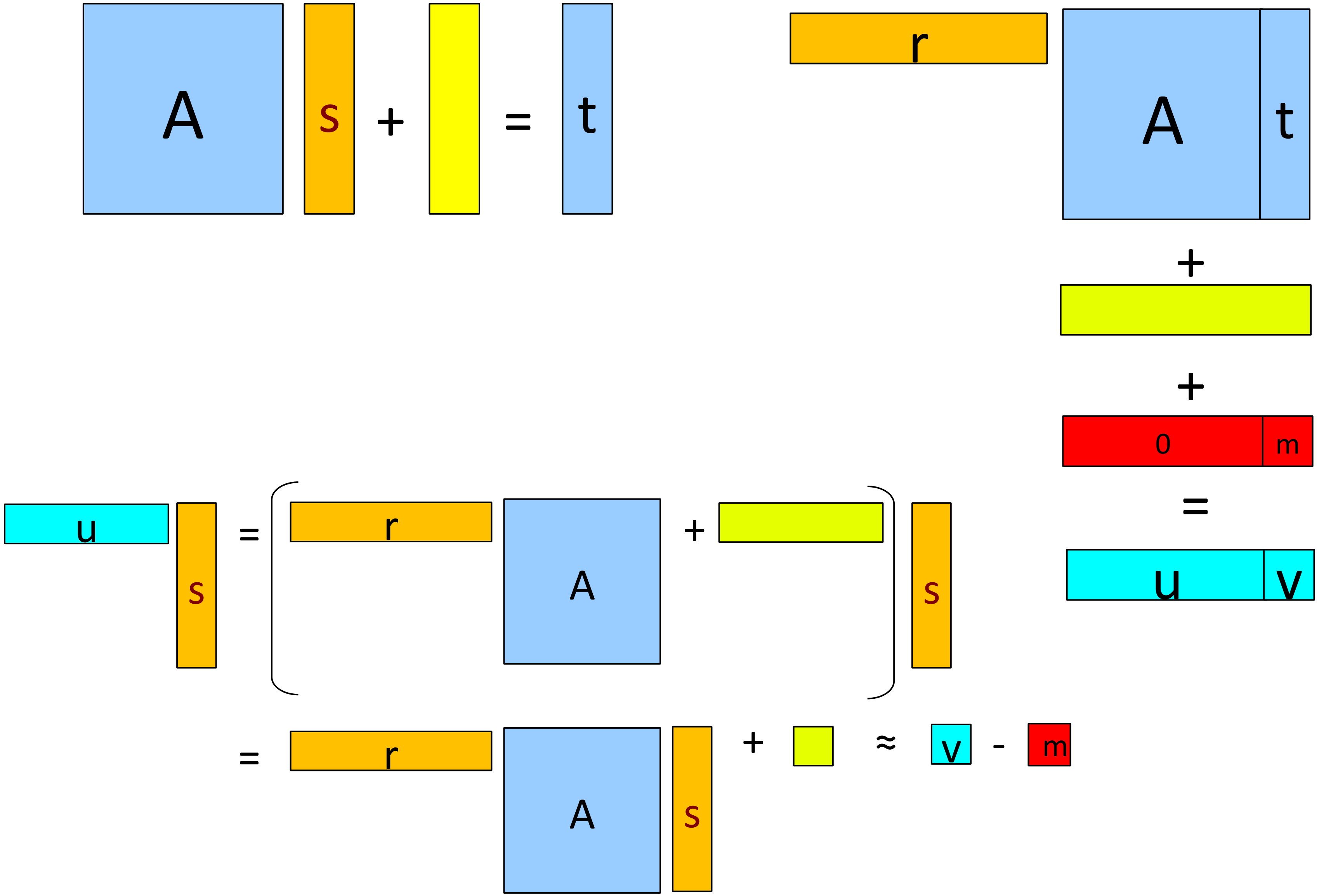


Is pseudo-random based on the hardness of the Learning with Errors Problem

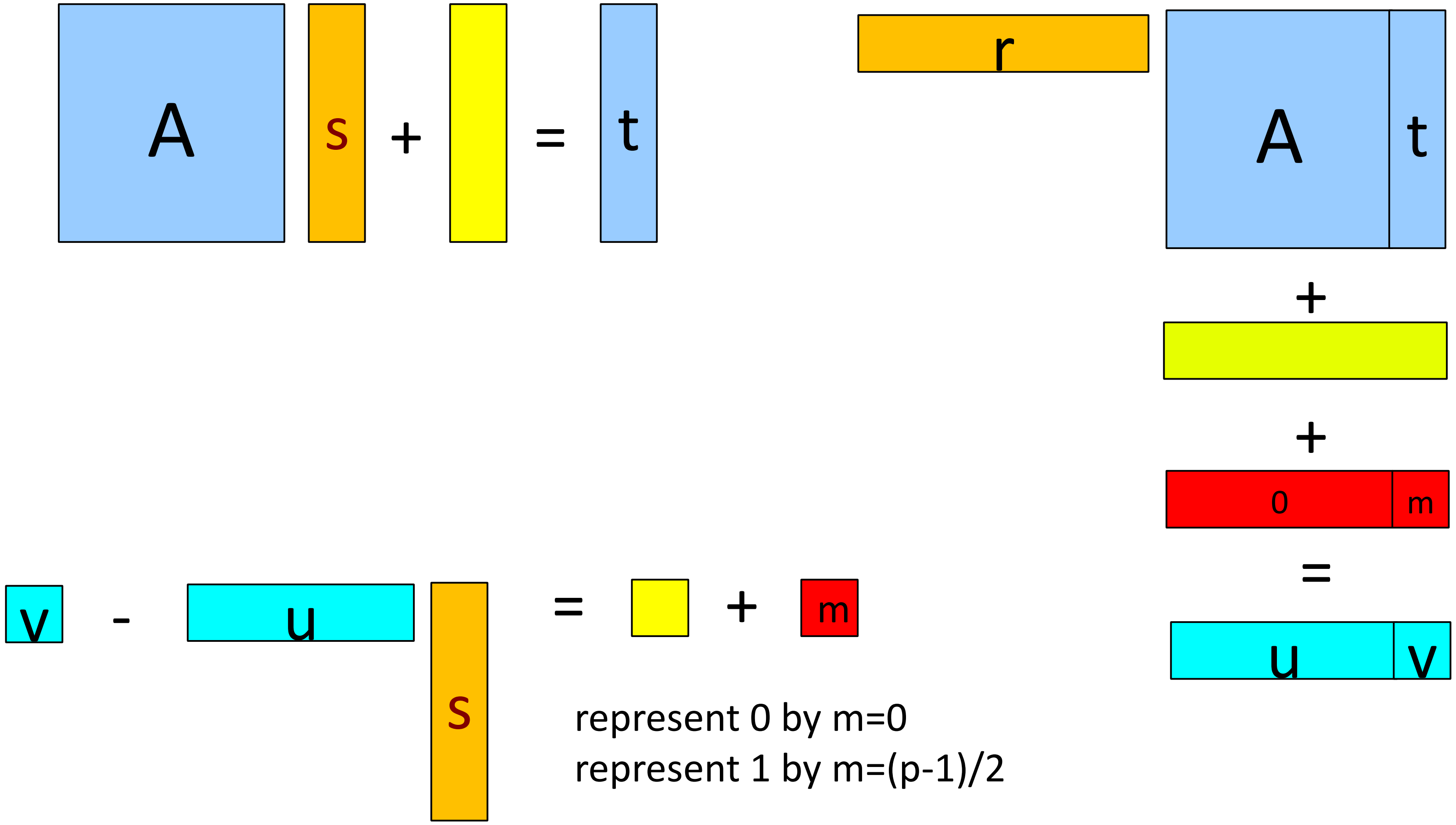
Encryption Scheme



Encryption Scheme



Encryption Scheme



Encrypts only 1 bit – large ciphertext expansion

1 bit requires n elements in Z_p

Decryption Error

Public Key: A, $t = As+e$

Ciphertext: $u = rA+e_1$, $v= rt + e_2 + m(p/2)$

Decryption: $v - us \ = r(As+e) + e_2 + m(p/2) – (rA+e_1)s$

$= re + e_2 + m(p/2) - e_1s$

Need the total error $re +e_2 - e_1s$ to be $< p/4$

(let's ignore e_2, since it's just an integer)

Say each coefficient of s, r, e, e_1, e_2 is uniformly random in {-1,0,1},

how do you make sure that the inequality is satisfied?

Decryption Error

- Can set p large enough so that re - e_1s to be < p/4 - 1
  - If the length of the vectors is n, then the maximum value is 2n
  - But intuitively, we expect the value to be around $\sqrt{n}$
  - So we will set p unnecessarily large

- Can use various inequalities (e.g. Chernoff, Hoeffding) and get closer to $\sqrt{n}$

- But we can do this much easier and more precisely

Decryption Error via Convolution

- Let's look at re = $\sum r_i \, e_i$ as the sum of n *independent* random variables
- What's the distribution of $r_i e_i$?
  - Pr[-1] = 2/9
  - Pr[0] = 5/9
  - Pr[1] = 2/9
- Write it as the polynomial $p(X)=(2/9)\ X^{-1} + (5/9)X^0 + (2/9)X^1$
- What's the distribution of $r_i e_i + r_j e_j$?
- Compute the product P(X)*P(X) and read off the coefficients!
- $\Pr[r_i e_i + r_j e_j = c]$ = the coefficient of X^c in P(X)*P(X)
- So $\Pr[re - e_1 s = c]$ = the coefficient of X^c in $P(X)^{2n}$

Problem Session

1. Implement the Encryption scheme:

    - p=257
    - Dimensions of A = 64 x 64
    - distribution of s,e,r,e_1,e_2 is Binomial: i.e. each coefficient is $b_0 + b_1 – b_2 – b_3$ (b_i are bits)

2. Write a script to compute the decryption error

Plan for the Week

1. Improve efficiency by working with polynomial rings
2. Number Theory Transform (like FFT)
3. Intro to Zero-Knowledge Proofs
4. Digital Signatures
5. Connection of these Constructions to Geometric Lattices

Supplementary Reading

https://github.com/VadimLyubash/LatticeTutorial/

(Today covered pages 1- 8)