Radboud University

Multivariate cryptography – Cryptanalysis techniques II

SLMath summer school:
Introduction to Quantum-Safe Cryptography (IBM Zurich)

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Recall the MinRank problem

MinRank $MR(n, m, r, M_1, \ldots, M_m)$
Input: $n, m, r \in \mathbb{N}$, and $M_1, \ldots, M_m \in \mathcal{M}_n(\mathbb{F}_q)$.
Question: Find – if any – a nonzero m-tuple $(\lambda_1, \ldots, \lambda_m) \in \mathbb{F}_q^m$ s.t.:

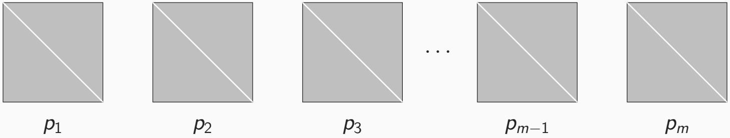
$$\text{Rank}\left(\sum_{i=1}^{m} \lambda_i M_i\right) \leqslant r.$$

[Courtois '01], [Buss & Shallit '99]

How do we use MinRank in cryptanalysis?

$\mathcal{P} = (p_1, p_2, \ldots, p_m)$ - public polynomials,

$\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_m$ - matrix representations of the coordinates of $\mathcal{P}$.



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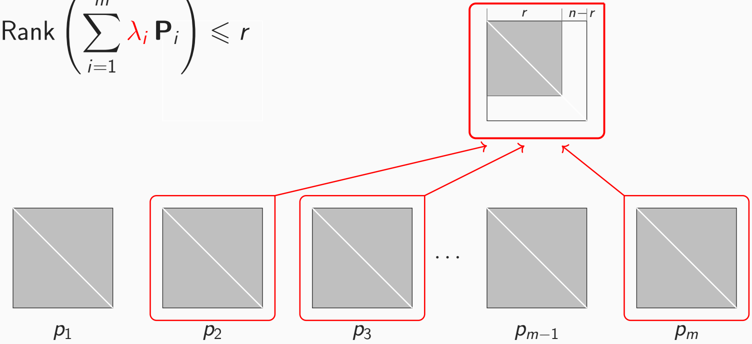

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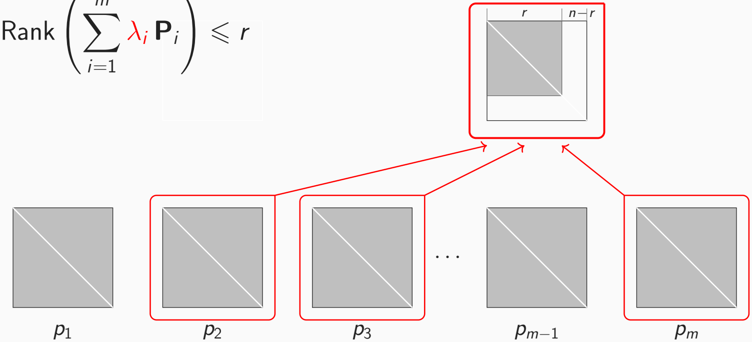
$$\text{Rank}\left(\sum_{i=1}^{m} {\color{red}\lambda_i}\, \mathbf{P}_i\right) \leqslant r$$



$\mathcal{S}$ is determined by

$$\text{Ker}\left(\sum_{i=1}^{m} {\color{red}\lambda_i}\, \mathbf{P}_i\right)$$

$\mathcal{T}$ is determined by

$$\langle {\color{red}(\lambda_1, \ldots, \lambda_m)} \rangle$$

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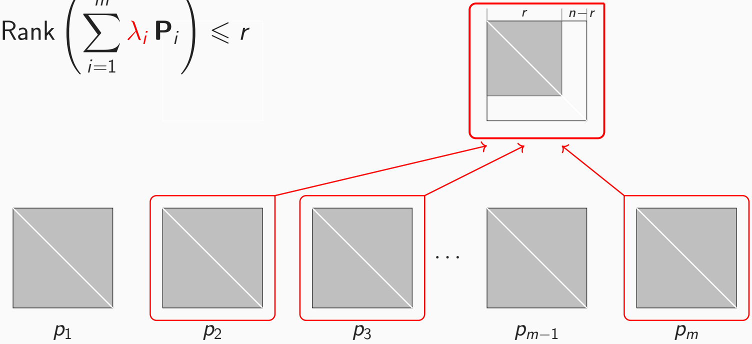

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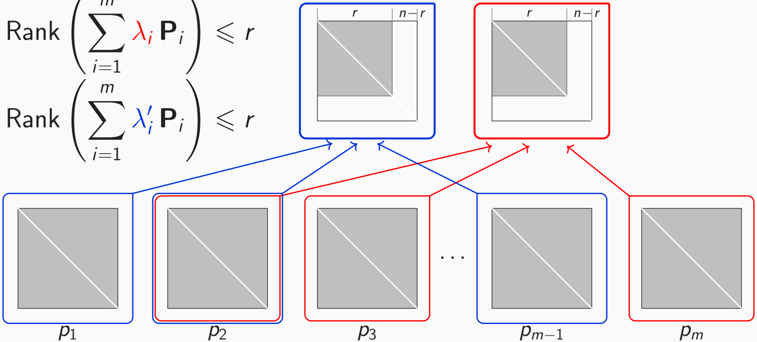

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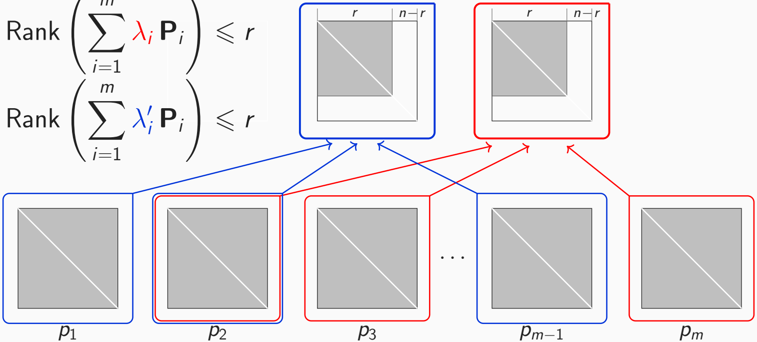

$\mathcal{S}$ is determined by

$$\text{Ker}\left(\sum_{i=1}^{m} \lambda_i \mathbf{P}_i\right) \cap \text{Ker}\left(\sum_{i=1}^{m} \lambda'_i \mathbf{P}_i\right)$$

$\mathcal{T}$ is determined by

$$\langle (\lambda_1, \ldots, \lambda_m), (\lambda_1, \ldots, \lambda_m) \rangle$$

Baby example UOV

Similar approach works for UOV, although it is not a result of "rank defect" (at leaset not so obvious)

$$f_1(x_1, \ldots, x_6) = x_1x_2 + x_2x_4 + x_3x_6 + x_4x_6 + x_5x_6 + x_6$$

$$f_2(x_1, \ldots, x_6) = x_1x_4 + x_3x_4 + x_3x_6 + x_4x_6 + x_6$$

$$f_3(x_1, \ldots, x_6) = x_2x_3 + x_3x_5 + x_2x_4 + x_2x_6 + x_4x_5 + x_1x_6 + x_4x_6 + x_5x_6$$

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$$\overline{S}': \begin{array}{l} x_4 \to x_4 + x_6 \\ x_2 \to x_2 + x_5 \end{array}$$

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$$\overline{S}': \begin{aligned} x_4 &\to x_4 + x_6 \\ x_2 &\to x_2 + x_5 \end{aligned}$$

After change of variables, we have separated (some) of the oil space (x_5, x_6) :

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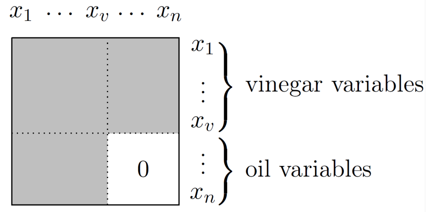
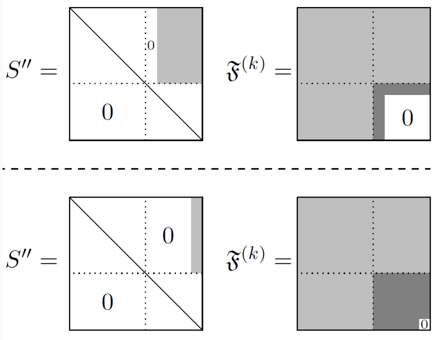
UOV

$$f_s(x) = \sum_{i \in V, j \in V} \gamma_{ij}^{(s)} x_i x_j + \sum_{i \in V, j \in O} \gamma_{ij}^{(s)} x_i x_j,$$

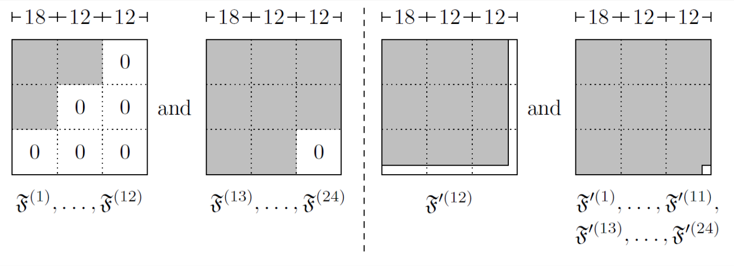


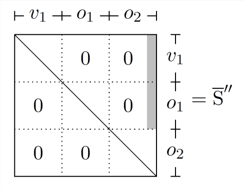


Good Keys for UOV

Rainbow partial revealing of structure ("good keys")

Rainbow before and after applying an input and output change of basis (separating a a good key)



Good key for Rainbow - 

Measuring linear spaces

Linear spaces of (n, m)-functions

- **Differential** of f: $\mathcal{D}_w f(x) = f(x+w) - f(x) - f(w) + f(0)$
- Linearity for (n, m) functions $f : \mathbb{F}_q^n \to \mathbb{F}_q^m$ defined already 1992 by Nyberg
- $w \in \mathbb{F}_q^n$ - **linear structure** of f if

$$\mathcal{D}_w f(x) = 0 \quad \forall\, x \in \mathbb{F}_q^n .$$

- **Linear space** of f - generated by the linear structures of f.

Quadratic form f**:** $\mathcal{D}_w f(x) = w^\intercal \mathbf{F} x$, for a symmetric matrix $\mathbf{F}$,

- Ker($\mathbf{F}$) - **linear space of** f**.**

[Nyberg92] **Quadratic** (n, m)**-function** f**:**

- Linearity - measured using the **smallest rank** r of any of the components $w^\intercal \cdot f$.

Maximum nonlinearity:

- **Bent functions** - Rank$(\mathbf{F}_w) = n$, even n, $m \leqslant n/2$,
- **Almost bent (AB) functions** - Rank$(\mathbf{F}_w) = n - 1$, odd $n = m$.

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Example 1:

f :

$$\begin{aligned} f_1 &= x_1x_2 + x_3 \\ f_2 &= x_1x_3 + x_2 + x_3 \\ f_3 &= x_2x_3 + x_1 + x_2 + x_3 \\ f_4 &= x_1x_2 \end{aligned}$$

$(1,0,0,1)^\intercal$ $\cdot f$ is linear

f' :

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Both have maximum linearity, but f' is linear on a larger space!

It is an important measure!

Oil & Vinegar maps

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$$f_1(x_1, x_2, x_3, x_4) = x_1x_3 + x_2x_4 + x_1x_2 + x_3$$
$$f_2(x_1, x_2, x_3, x_4) = x_2x_3 + x_1x_4 + x_2x_4 + x_3$$

f is linear on the oil subspace (when you fix the vinegar variables)!

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Boura and Canteaut FSE13:

f is said to be **(s,t)–linear** if there exist linear subspaces
$V \subset \mathbb{F}_q^n$ with $\mathrm{Dim}(V) = s$, $W \subset \mathbb{F}_q^m$ with $\mathrm{Dim}(W) = t$, s.t.

$$\forall\, w \in W, \;\; w^{\intercal} \cdot f \text{ is linear on all cosets of } V.$$

(s,t)–linearity of quadratic (n,m) function f

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- f_W corresponding to all $w^{\intercal} \cdot f$, $w \in W$ can be written as

$$f_W(x,y) = M(x) \cdot y + G(x)$$

where $\mathbb{F}_q^n = U \oplus V$, $G : U \to \mathbb{F}_q^t$ and $M(x)$ is a $t \times s$ matrix with rows - components of linear functions over U.

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- for $w \in W$, $\mathcal{D}_a w^\intercal \cdot f(b) = 0$, $\forall\ a, b \in V$.
- for $w \in W$, $f_W(0,y) = M(0)\cdot y + G(0) = 0, \forall\ (0,a) \in V$ - **all components in W vanish on the V space**

Example:

$$f_1(x_1, x_2, x_3, x_4) = x_1x_3 + x_2x_4 + x_1x_2 + x_3$$
$$f_2(x_1, x_2, x_3, x_4) = x_2x_3 + x_1x_4 + x_2x_4 + x_3$$

f is $(2,2)$–linear,
$V = \langle (0,0,1,0), (0,0,0,1) \rangle$, $W = \langle (1,0), (0,1) \rangle$

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f is $(3,2)$–linear,
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$\mathcal{D}_a f(b) = 0, \quad \forall\ a, b$ in the oil space O.

$f(0, a) = 0, \forall\ a \in O$ - **the oil and vinegar map vanishes on the oil space!**

Basis for the new definition of UOV [Beullens21]

A consequence? - **Reconciliation Attack [Ding et al.]**

In a nutshel: **Recover** (s, m) **linearity** of the public $\mathcal{P} : \mathbf{P}_1, \ldots, \mathbf{P}_m$

Solve:

$$\begin{aligned} x^{(j)}\mathbf{P}_i x^{(k)} &= 0,\ i \in \{1, ..., m\},\ j, k \in \{1, ..., s\}, j < k \\ x^{(k)}\widetilde{\mathbf{P}}_i x^{(k)} &= 0,\ i \in \{1, ..., m\},\ k \in \{1, ..., s\}, \end{aligned}$$

in the unknown basis vectors $x^{(j)}$ of the oil space O,

where $\mathbf{P}_i := \widetilde{\mathbf{P}}_i + \widetilde{\mathbf{P}}_i^\mathsf{T}$.

Back to UOV - what does the above mean?

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Reconciliation attack [Ding et al.]

As given in [SG14]:

1. Solve the quadratic

$$
\begin{aligned}
x^{(j)}\mathbf{P}_i x^{(k)} &= 0,\ \ i \in \{1, \dots, m\},\ \ j,k \in \{1, \dots, c\}, j < k \\
x^{(k)}\widetilde{\mathbf{P}}_i x^{(k)} &= 0,\ \ i \in \{1, \dots, m\},\ \ k \in \{1, \dots, c\},
\end{aligned}
$$

   in the unknown basis vectors $x^{(k)}$ of the space O.

   [$m\binom{c+1}{2}$ **quadratic and bilinear equations** $(n-m)c$ **variables**
   We must choose c s.t. $m\binom{c+1}{2} \geq (n-m)c$ **(typically at least 2)**]

2. Then solve the linear

$$
x^{(j)}\mathbf{P}_i x^{(k)} = 0,\ \ i \in \{1, \dots, m\}, j \in \{1, \dots, c\}, k \in \{c+1, \dots, m\}, j < k
$$

   in the unknown basis vectors $x^{(k)}$ of the oil space O.

   [For first k, mc **linear equations** $(n-m)$ **variables**
   Works if $m(c+1) \geq n$,
   otherwise plug in in step 1 and solve easier quadratic system]

Making the reconciliation attack practical

Important about the attack:

- If c taken big enough in the first step, second step is always polynomial
- **First step is the expensive one**
- **Questions:**
  - Can we have a polynomial second step for smaller c?
  - **Yes, only one vector seems to be enough!**
  - Can we find easier (than step 1) vectors in the oil space?
  - **Yes, intersection attack!**

One oil vector breaks UOV!

- Shown in [Aulbach, Campos, Krämer, S, Stöttinger '23]
- Simpler view in [Pébereau'24]
  - Assume $n \leq 3m$
  - Assume an oil vector o is known
  - Recall that $\boxed{\mathcal{D}_o f(b) = 0, \quad \forall\ b \text{ in the oil space } O.}$

    so the oil space O lives in the kernel of the differential $\mathcal{D}_o$

    $$|\mathrm{Ker}(\mathcal{D}_o)| = n - m$$

  - Restrict the public key to $\mathrm{Ker}(\mathcal{D}_o)$ using a basis matrix $\mathbf{S}_{\mathrm{Ker}}$

    $$\mathcal{P}_{|\,\mathrm{Ker}(\mathcal{D}_o)} = \mathcal{P} \circ \mathbf{S}_{\mathrm{Ker}}$$

  - Obtain a $(n - m, m)$ UOV instance
    - Unknown oil space O' can be found by Kipnis-Shamir attack '98 (becomes polynomial)
    - Alternatively, use Step 2 of reconciliation attack for $c = 1$ (becomes polynomial)
  - Go back to original UOV instance
    - Basis of unknown oil space $\mathbf{B}_O = \mathbf{S}_{\mathrm{Ker}} \cdot \mathbf{B}_{O'}$

Finding one oil vector: enhancement of Kipnis-Shamir attack '98

- Kipnis-Shamir attack '98 - Broke Oil & Vinegar by Patarin ($n = 2m$)
- Recall that $\mathcal{D}_o f(b) = 0, \quad \forall\ b$ in the oil space O.
- In matrix form

$$
\begin{aligned}
o^{(j)} \mathbf{P}_i o^{(k)} &= 0,\ i \in \{1, ..., m\},\ j, k \in \{1, ..., m\} \\
\mathbf{P}_i \cdot O &\subset O^{\perp}
\end{aligned}
$$

- $|\mathbf{P}_i \cdot O| = m, \ \left|O^{\perp}\right| = n - m$
- $|\mathbf{P}_i \cdot O \cap \mathbf{P}_j \cdot O| \geq |\mathbf{P}_i \cdot O| + |\mathbf{P}_j \cdot O| - \left|O^{\perp}\right| = 3m - n$

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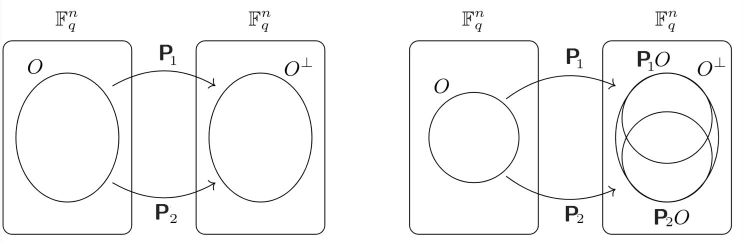

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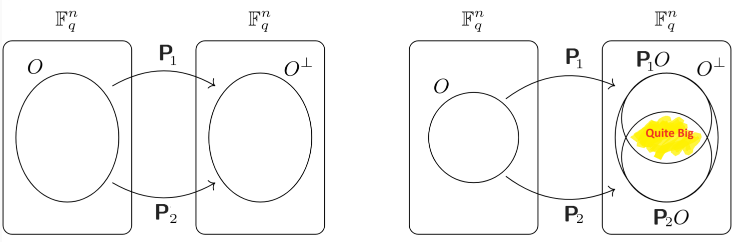

Intersection attack [Beullens '20]

- Focus on $n < 3m$
- **We want to find x in the intersection $\mathbf{P}_i \cdot O \cap \mathbf{P}_j \cdot O$**
- But then **$\mathbf{P}_i^{-1}x \in O$ and $\mathbf{P}_j^{-1}x \in O$ are two oil vectors**
- We can do the reconciliation attack but on steroids!
  - Fix $3m - n$ coordinates of x and solve the quadratic system

$$
\begin{aligned}
(\mathbf{P}_1^{-1}x)^\top \mathbf{P}_i \mathbf{P}_2^{-1}x &= 0, \; i \in \{1, ..., m\} \\
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  - $3m$ equations and $2n - 3m$ variables
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Now, let's think again about our measures of linearity

Better said, let's take a different perspective...

So far we considered m symmetric matrices representing our polynomials.

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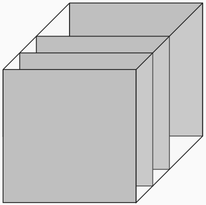

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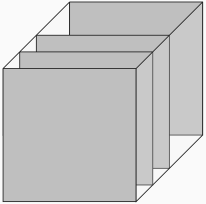

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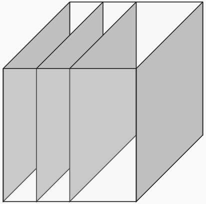

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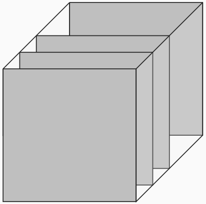

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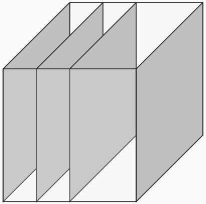

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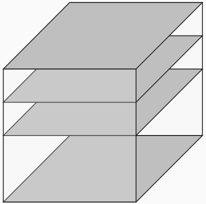

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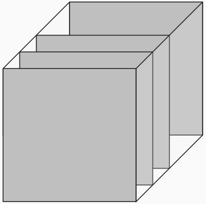

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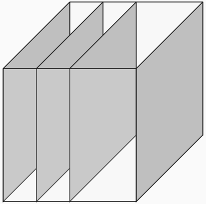

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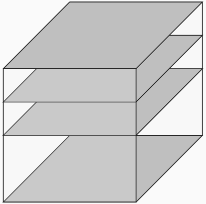

- This is different **tensor view**, but the same object!
- Instead of array of two-dimensional matrices, we look at it as a three-dimensional qube!

How does this change perspective?

Recall, UOV has an important hidden linear spaces (the oil space)...

But no rank defects!

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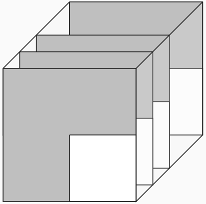

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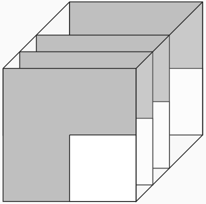

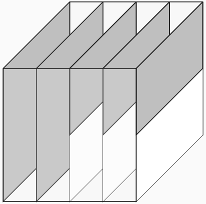

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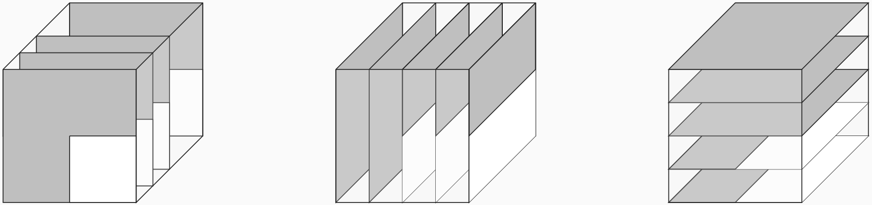

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- We can use (Rectangular) MinRank!
- Beullens '22 used it to improve the analysis on UOV and Rainbow
- **Important takeaway:** The two types of important linear spaces can be characterized in the same way!

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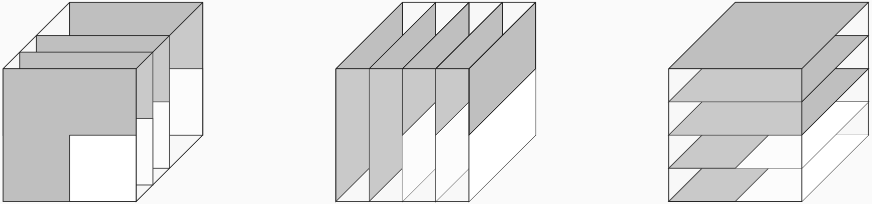

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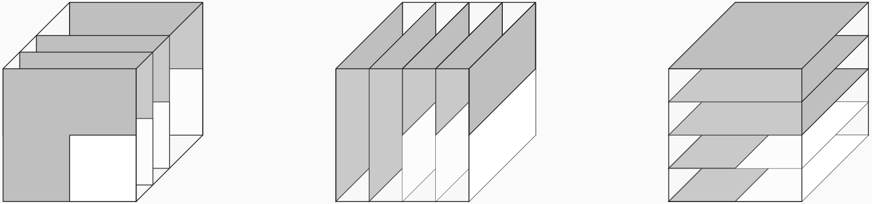

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Equivalent keys used wrong

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- **Now a finalist**
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  - UOV map
  - Sparse polynomials to reduce key size - only v coefficients per polynomial
  - Four variants with different level of sparsness
  - Equivalent keys technique to reduce key size
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MQ-Sign Central map

$$\mathcal{F}_V^{(1)} = \sum_{i=1}^{v} \gamma_i^{(1)} x_i x_{(i \bmod v)+1} \quad \rightarrow \quad \mathbf{F}_V^{(1)} = \begin{pmatrix} 0 & \gamma_1^{(1)} & 0 & \cdots & 0 \\ 0 & 0 & \gamma_2^{(1)} & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \gamma_{v-1}^{(1)} \\ \gamma_v^{(1)} & 0 & 0 & \cdots & 0 \end{pmatrix}$$

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Derive Linear Equations from Key Equation

Recall that the key equation $\mathcal{P} = \mathcal{F} \circ \mathbf{S}$ translates to the matrix equations $\mathbf{P^{(k)}} = \mathbf{S}^\top \mathbf{F^{(k)}} \mathbf{S}$, i.e.

$$\begin{pmatrix} \mathbf{P}_1^{(k)} & \mathbf{P}_2^{(k)} \\ \mathbf{0} & \mathbf{P}_4^{(k)} \end{pmatrix} = \mathtt{Upper}\left(\begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{S}_1^\top & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{F}_1^{(k)} & \mathbf{F}_2^{(k)} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{I} & \mathbf{S}_1 \\ \mathbf{0} & \mathbf{I} \end{pmatrix} \right)$$

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$$= \begin{pmatrix} \mathbf{F}_1^{(k)} & (\mathbf{F}_1^{(k)} + \mathbf{F}_1^{(k)\top})\mathbf{S}_1 + \mathbf{F}_2^{(k)} \\ \mathbf{0} & \mathtt{Upper}\ (\mathbf{S}_1^\top \mathbf{F}_1^{(k)} \mathbf{S}_1 + \mathbf{S}_1^\top \mathbf{F}_2^{(k)}) \end{pmatrix}.$$

From the two upper blocks, as previous, we obtain the equations

$$\begin{aligned} \mathbf{P}_1^{(k)} &= \mathbf{F}_1^{(k)} \quad \text{and} \\ \mathbf{P}_2^{(k)} &= (\mathbf{P}_1^{(k)} + \mathbf{P}_1^{(k)\top})\mathbf{S}_1 + \mathbf{F}_2^{(k)}. \end{aligned}$$

- The second is a system of linear equations in the entries of the secret $\mathbf{S_1}$
- Still, not possible to determine them, due to the secret coefficients in $\mathbf{F}_2^{(k)}$

In two variants of MQ-Sign, the coefficients in $\mathbf{F}_2^{(k)}$ are chosen sparsely.

This removes unknown variables from the system

$$\mathbf{P}_2^{(k)} = (\mathbf{P}_1^{(k)} + \mathbf{P}_1^{(k)\top})\mathbf{S}_1 + \mathbf{F}_2^{(k)}.$$

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- Collect linear equations for all $k \in \{1, \ldots, m\}$ polynomials.
- Obtain system of $mv(m-1)$ equations in vm variables (can be divided into subsystems).
- Once $\mathbf{S}$ is known, the central polynomials can efficiently be found.

The constructed key is actually not equivalent to a UOV key, it is weaker!

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