Multivariate Fiat-Shamir signatures

SLMath summer school:
Introduction to Quantum-Safe Cryptography (IBM Zurich)

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Recall the MQ problem from last time

Computational MQ problem

Given: m multivariate polynomials $p_1, p_2, \ldots, p_m \in \mathbb{F}_q[x_1, \ldots, x_n]$ of degree 2

Find: (if any) a vector $(u_1, \ldots, u_n) \in \mathbb{F}_q^n$ such that

$$\begin{cases} p_1(u_1, \ldots, u_n) = & 0 \\ p_2(u_1, \ldots, u_n) = & 0 \\ \quad \ldots & \\ p_m(u_1, \ldots, u_n) = & 0 \end{cases}$$

- Recall also that traditionally MQ schemes are ad-hoc
  - the hard problem is not the MQ problem, and not only the MQ problem
- What does it take to get a provably secure MQ scheme?
  - **MQDSS**: first signature with (lossy) ROM reduction to MQ
  - **SOFIA**: first signature with (lossy) QROM reduction to MQ

Some brainstorming in Sofia with Andy and Peter

- Lack of provable MQ signature
- Inefficient signatures from 3-pass IDS [Sakumoto et al. '11]
  - big soundness error (2/3)

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  - smaller soundness error ($\frac{q+1}{2q}$ over $\mathbb{F}_q$) $\Rightarrow$ smaller signatures
  - FS transform for 5-pass already available [El Yousfi '12]
    - loose reduction in the ROM (as for 3-pass [Pointcheval & Stern '96])

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Canonical Identification Schemes

$\mathcal{P}$, sk		$\mathcal{V}$, pk
$\mathsf{com} \leftarrow_R \mathcal{P}_0(\mathsf{sk})$	$\xrightarrow{\mathsf{com}}$	
		$\mathsf{ch} \leftarrow_R \mathsf{ChS}(1^k)$
	$\xleftarrow{\mathsf{ch}}$	
$\mathsf{resp} \leftarrow \mathcal{P}_1(\mathsf{sk}, \mathsf{com}, \mathsf{ch})$	$\xrightarrow{\mathsf{resp}}$	
		$b \leftarrow \mathsf{Vf}(\mathsf{pk}, \mathsf{com}, \mathsf{ch}, \mathsf{resp})$

Informally:

(1) Prover commits to some (randomized) value derived from sk

(2) Verifier picks a challenge ‘ch’

(3) Prover computes response ‘resp’

(4) Verifier checks if response matches challenge

Properties of Canonical 3-pass IDS

$$
\begin{array}{lcl}
\mathcal{P}, \mathsf{sk} & & \mathcal{V}, \mathsf{pk} \\
\hline
\mathsf{com} \leftarrow_R \mathcal{P}_0(\mathsf{sk}) & \xrightarrow{\mathsf{com}} & \\
 & & \mathsf{ch} \leftarrow_R \mathsf{ChS}(1^k) \\
 & \xleftarrow{\mathsf{ch}} & \\
\mathsf{resp} \leftarrow \mathcal{P}_1(\mathsf{sk}, \mathsf{com}, \mathsf{ch}) & & \\
 & \xrightarrow{\mathsf{resp}} & \\
 & & b \leftarrow \mathsf{Vf}(\mathsf{pk}, \mathsf{com}, \mathsf{ch}, \mathsf{resp})
\end{array}
$$

- Special soundness
  There exists knowledge extractor $\mathcal{K}$ s.t. given two valid transcripts:

  $$\mathsf{trans} = (\mathsf{com}, \mathsf{ch}, \mathsf{resp}),\ \mathsf{trans}' = (\mathsf{com}, \mathsf{ch}', \mathsf{resp}'), \quad \mathsf{ch} \neq ch',$$

  extracts the secret sk with non-negligible probability
- (statistical) Honest-Verifier Zero-Knowledge
  There exists a PPT algorithm $\mathcal{S}$, called the simulator, such that the statistical distance between the real transcript and the simulated transcript is negligible in k.

The Fiat-Shamir transform

IDS

$\mathcal{P}$		$\mathcal{V}$
$\mathsf{com} \leftarrow \mathcal{P}_0\ (\mathsf{sk})$	$\xrightarrow{\mathsf{com}}$	
	$\xleftarrow{\mathsf{ch}}$	$\mathsf{ch} \leftarrow_R \mathsf{ChS}\ (1^k)$
$\mathsf{resp} \leftarrow \mathcal{P}_1\ (\mathsf{sk}, \mathsf{com}, \mathsf{ch})$	$\xrightarrow{\mathsf{resp}}$	
		$b \leftarrow \mathsf{Vf}\ (\mathsf{pk}, \mathsf{com}, \mathsf{ch}, \mathsf{resp})$

The Fiat-Shamir transform

IDS

$$
\begin{array}{lcr}
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\hline
\mathsf{com} \leftarrow \mathcal{P}_0{}^{\color{red}r}(\mathsf{sk}) & \xrightarrow{\quad \mathsf{com} \quad} & \\
 & & \mathsf{ch} \leftarrow_R \mathsf{ChS}^{\color{red}r}(1^k) \\
 & \xleftarrow{\quad \mathsf{ch} \quad} & \\
\mathsf{resp} \leftarrow \mathcal{P}_1{}^{\color{red}r}(\mathsf{sk}, \mathsf{com}, \mathsf{ch}) & \xrightarrow{\quad \mathsf{resp} \quad} & \\
 & & b \leftarrow \mathsf{Vf}^{\color{red}r}(\mathsf{pk}, \mathsf{com}, \mathsf{ch}, \mathsf{resp})
\end{array}
$$

The Fiat-Shamir transform

IDS

$\mathcal{P}^r$		$\mathcal{V}^r$
$\mathsf{com} \leftarrow \mathcal{P}_0{}^r(\mathsf{sk})$	$\xrightarrow{\mathsf{com}}$	
$\mathsf{ch} \leftarrow_R \mathsf{ChS}^r(1^k)$	$\xleftarrow{\mathsf{ch}}$	
$\mathsf{resp} \leftarrow \mathcal{P}_1{}^r(\mathsf{sk}, \mathsf{com}, \mathsf{ch})$	$\xrightarrow{\mathsf{resp}}$	
		$b \leftarrow \mathsf{Vf}^r(\mathsf{pk}, \mathsf{com}, \mathsf{ch}, \mathsf{resp})$

$\downarrow \quad \downarrow \quad \downarrow$

FS signature

Signer
$\mathsf{com} \leftarrow \mathcal{P}_0(\mathsf{sk})$
$\mathsf{ch} \leftarrow H(m, \mathsf{com})$
$\mathsf{resp} \leftarrow \mathcal{P}_1(\mathsf{sk}, \mathsf{com}, \mathsf{ch})$
output : $\sigma = (\mathsf{com}, \mathsf{resp})$

Verifier
$\mathsf{ch} \leftarrow H(m, \mathsf{com})$
$b \leftarrow \mathsf{Vf}(\mathsf{pk}, \mathsf{com}, \mathsf{ch}, \mathsf{resp})$
output : b

Security of FS signatures [Pointcheval & Stern '96]

IDS		**FS signature**
Special soundness	$\longrightarrow$	Secure under key-only attack

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$$\mathsf{trans} = (\mathsf{com}, \mathsf{ch}, \mathsf{resp})$$

$$\mathsf{trans}' = (\mathsf{com}, \mathsf{ch}', \mathsf{resp}')$$

IDS | **FS signature**

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Fork.lemma

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$\xrightarrow{\mathcal{K}}$ sk

Security of FS signatures [Pointcheval & Stern '96]

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$+$	$\searrow$	
HVZK	$\longrightarrow$	**eu-cma secure**

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$\mathcal{O} \leftrightarrow \mathcal{B}(\mathrm{pk}) \leftrightarrow \mathcal{S}$

$\mathcal{B}(\mathrm{pk}) \rightsquigarrow \sigma = (\mathrm{com}, \mathrm{ch}, \mathrm{resp})$

$\Rightarrow \quad \mathcal{A}(pk) \rightsquigarrow \mathrm{sk}$

Canonical 5-pass IDS

$$
\begin{array}{lcl}
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\mathsf{com} \leftarrow \mathcal{P}_0(\mathsf{sk}) & \xrightarrow{\quad \mathsf{com} \quad} & \\
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\mathsf{resp}_1 \leftarrow \mathcal{P}_1(\mathsf{sk}, \mathsf{com}, \mathsf{ch}_1) & \xrightarrow{\quad \mathsf{resp}_1 \quad} & \\
 & \xleftarrow{\quad \mathsf{ch}_2 \quad} & \mathsf{ch}_2 \leftarrow_R \mathsf{ChS}_2(1^k) \\
\mathsf{resp}_2 \leftarrow \mathcal{P}_2(\mathsf{sk}, \mathsf{com}, \mathsf{ch}_1, \mathsf{resp}_1, \mathsf{ch}_2) & \xrightarrow{\quad \mathsf{resp}_2 \quad} & \\
 & & b \leftarrow \mathsf{Vf}(\mathsf{pk}, \mathsf{com}, \mathsf{ch}_1, \mathsf{resp}_1, \mathsf{ch}_2, \mathsf{resp}_2)
\end{array}
$$

The Fiat-Shamir transform on 5-pass IDS

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$\mathsf{resp}_1 \leftarrow \mathcal{P}_1(\mathsf{sk}, \mathsf{com}, \mathsf{ch}_1)$	$\xrightarrow{\mathsf{resp}_1}$	
	$\xleftarrow{\mathsf{ch}_2}$	$\mathsf{ch}_2 \leftarrow_R \mathsf{ChS}_2(1^k)$
$\mathsf{resp}_2 \leftarrow \mathcal{P}_2(\mathsf{sk}, \mathsf{com}, \mathsf{ch}_1, \mathsf{resp}_1, \mathsf{ch}_2)$	$\xrightarrow{\mathsf{resp}_2}$	
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$\downarrow \; \downarrow \; \downarrow$

FS signature

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$\mathsf{com} \leftarrow \mathcal{P}_0(\mathsf{sk})$

$\mathsf{ch}_1 \leftarrow H_1(m, \mathsf{com})$

$\mathsf{resp}_1 \leftarrow \mathcal{P}_1(\mathsf{sk}, \mathsf{com}, \mathsf{ch}_1)$

$\mathsf{ch}_2 \leftarrow H_2(m, \mathsf{com}, \mathsf{ch}_1, \mathsf{resp}_1)$

$\mathsf{resp}_2 \leftarrow \mathcal{P}_2(\mathsf{sk}, \mathsf{com}, \mathsf{ch}_1, \mathsf{resp}_1, \mathsf{ch}_2)$

output : $\sigma = (\mathsf{com}, \mathsf{resp}_1, \mathsf{resp}_2)$

Verifier

$\mathsf{ch}_1 \leftarrow H_1(m, \mathsf{com})$

$\mathsf{ch}_2 \leftarrow H_2(m, \mathsf{com}, \mathsf{ch}_1, \mathsf{resp}_1)$

$b \leftarrow \mathsf{Vf}(\mathsf{pk}, \mathsf{com}, \mathsf{ch}_1, \mathsf{resp}_1, \mathsf{ch}_2, \mathsf{resp}_2)$

output : b

Sakumoto-Shirai-Hiwatari 5-pass IDS

$\mathcal{P}(\mathbf{F}, \mathbf{v}, \mathbf{s})$		$\mathcal{V}(\mathbf{F}, \mathbf{v})$
$\mathbf{r}_0, \mathbf{t}_0 \leftarrow_R \mathbb{F}_q^n, \mathbf{e}_0 \leftarrow_R \mathbb{F}_q^m$		
$\mathbf{r}_1 \leftarrow \mathbf{s} - \mathbf{r}_0$		
$c_0 \leftarrow Com(\mathbf{r}_0, \mathbf{t}_0, \mathbf{e}_0)$		
$c_1 \leftarrow Com(\mathbf{r}_1, \mathbf{G}(\mathbf{t}_0, \mathbf{r}_1) + \mathbf{e}_0)$	$\xrightarrow{(c_0, c_1)}$	
	$\xleftarrow{\alpha}$	$\alpha \leftarrow_R \mathbb{F}_q$
$\mathbf{t}_1 \leftarrow \alpha \mathbf{r}_0 - \mathbf{t}_0$		
$\mathbf{e}_1 \leftarrow \alpha \mathbf{F}(\mathbf{r}_0) - \mathbf{e}_0$	$\xrightarrow{\text{resp}_1 = (\mathbf{t}_1, \mathbf{e}_1)}$	
	$\xleftarrow{\text{ch}_2}$	$\text{ch}_2 \leftarrow_R \{0, 1\}$
If $\text{ch}_2 = 0$, $\text{resp}_2 \leftarrow \mathbf{r}_0$		
Else $\text{resp}_2 \leftarrow \mathbf{r}_1$	$\xrightarrow{\text{resp}_2}$	
		If $\text{ch}_2 = 0$, Parse $\text{resp}_2 = \mathbf{r}_0$, check $c_0 \stackrel{?}{=} Com(\mathbf{r}_0, \alpha \mathbf{r}_0 - \mathbf{t}_1, \alpha \mathbf{F}(\mathbf{r}_0) - \mathbf{e}_1)$
		Else Parse $\text{resp}_2 = \mathbf{r}_1$, check $c_1 \stackrel{?}{=} Com(\mathbf{r}_1, \alpha(\mathbf{v} - \mathbf{F}(\mathbf{r}_1)) - \mathbf{G}(\mathbf{t}_1, \mathbf{r}_1) - \mathbf{e}_1)$

Sakumoto-Shirai-Hiwatari IDS

- Smaller soundness error ($\frac{q+1}{2q}$ over $\mathbb{F}_q$) $\Rightarrow$ **smaller signatures**
- Key technique: **cut-and-choose for** MQ

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- Bilinear map $\mathbf{G}(\mathbf{x}, \mathbf{y}) = \mathbf{F}(\mathbf{x} + \mathbf{y}) - \mathbf{F}(\mathbf{x}) - \mathbf{F}(\mathbf{y})$
  - Split $\mathbf{s}$ and $\mathbf{F}(\mathbf{s})$ into $\mathbf{r}_0, \mathbf{r}_1$ and $\mathbf{F}(\mathbf{r}_0), \mathbf{F}(\mathbf{r}_1)$
    - Since then $\mathbf{s} = \mathbf{r}_0 + \mathbf{r}_1 \Rightarrow \mathbf{F}(\mathbf{s}) = \mathbf{G}(\mathbf{r}_0, \mathbf{r}_1) + \mathbf{F}(\mathbf{r}_0) + \mathbf{F}(\mathbf{r}_1)$
  - Split $\mathbf{r}_0$ and $\mathbf{F}(\mathbf{r}_0)$ further into $\mathbf{t}_0, \mathbf{t}_1$ resp. $\mathbf{e}_0, \mathbf{e}_1$

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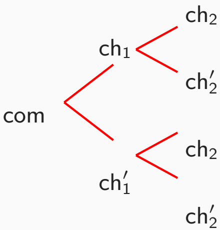
  - Split $\mathbf{r}_0$ and $\mathbf{F}(\mathbf{r}_0)$ further into $\mathbf{t}_0, \mathbf{t}_1$ resp. $\mathbf{e}_0, \mathbf{e}_1$
  - $\mathbf{r_0} = \mathbf{t_0} + \mathbf{t_1}$
  - $\mathbf{F}(\mathbf{r_0}) = \mathbf{e_0} + \mathbf{e_1}$
- Using bilinearity, $\mathbf{v} = (\mathbf{G}(\mathbf{t_0}, \mathbf{r_1}) + \mathbf{e_0}) + (\mathbf{F}(\mathbf{r_1}) + \mathbf{G}(\mathbf{t_1}, \mathbf{r_1}) + \mathbf{e_1})$
- Result: reveal either $\mathbf{r}_0$ or $\mathbf{r}_1$, and $(\mathbf{t}_1, \mathbf{e}_1)$
- **Zero knowledge property satisfied**

Sakumoto et al. 5-pass IDS

The extractor $\mathcal{K}$ **needs 4 valid transcripts!**

$$(\mathsf{com}, \mathsf{ch}_1, \mathsf{resp}_1, \mathsf{ch}_2, \mathsf{resp}_2)$$
$$(\mathsf{com}, \mathsf{ch}_1, \mathsf{resp}_1, \mathsf{ch}_2', \mathsf{resp}_2')$$
$$(\mathsf{com}, \mathsf{ch}_1', \mathsf{resp}_1', \mathsf{ch}_2, \mathsf{resp}_2'')$$
$$(\mathsf{com}, \mathsf{ch}_1', \mathsf{resp}_1', \mathsf{ch}_2', \mathsf{resp}_2''')$$



Sakumoto et al. 5-pass IDS

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$$(\mathsf{com}, \mathsf{ch}_1, \mathsf{resp}_1, \mathsf{ch}_2, \mathsf{resp}_2)$$
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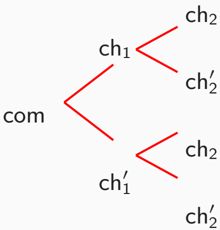
$$(\mathsf{com}, \mathsf{ch}_1', \mathsf{resp}_1', \mathsf{ch}_2', \mathsf{resp}_2''')$$



- Focus attention on 5-pass IDS with second challenge space $|\mathsf{ChS}_2| = 2$
  - Sakumoto et al. 5-pass IDS is such
  - Most in the literature are such

Next step ...

What is the problem with FS proof in the QROM?

- We need **to see** the signature σ before rewinding
- We need **to see** the oracle inputs
- Seeing (measuring) destroys the quantum state
- The proof fails terribly

Next step ...MQ signatures in the QROM

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A solution: Unruh transform [Unruh '14] adapted for $q2$ IDS

- Online extractability
- We can extract the witness without rewinding
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We proposed SOFIA - MQ signature secure in the QROM

MQDSS vs SOFIA

	Sec.	q	n $(= m)$	r	pk (bytes)	sk (bytes)	Signature (bytes)
MQDSS-31-64 (AC '16)	128 (ROM)	31	48	269	72	64	40952
SOFIA-4-128 (PKC '18)	128 (QROM)	4	128	438	64	32	126176

- SOFIA still comparable to Picnic (with QROM proof),
- but much slower than SPHINCS + and lattice based schemes

NIST parameter sets MQDSS

	Sec. cat.	q	n $(= m)$	r	pk (bytes)	sk (bytes)	Signature (bytes)
MQDSS-31-48 (Round 2)	1-2	31	48	135	46	16	20854
MQDSS-31-64 (Round 2)	3-4	31	64	202	64	24	43728

An attack on MQDSS

- August 2019, Daniel Kales and Greg Zaverucha - forgery in approx. 2^{95} hash calls for MQDSS-31-48
- Can be mitigated by $\approx 1.4\times$(number of rounds)
- **Proof still valid!**
  - Attack is result of not taking into account non-tightness of proof for choosing parameters

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- Can be mitigated by $\approx 1.4\times$(number of rounds)
- **Proof still valid!**
  - Attack is result of not taking into account non-tightness of proof for choosing parameters
- **New parameters after attack (estimate):**

	Sec. cat.	q	n	r	pk	sk	Signature
MQDSS-31-48 (new)	1-2	31	48	184	46B	16B	28400B
MQDSS-31-48 (Round 2)	1-2	31	48	135	46**B**	16**B**	20854**B**
MQDSS-31-64 (new)	3-4	31	64	277	64B	24B	59928B
MQDSS-31-64 (Round 2)	3-4	31	64	202	64**B**	24**B**	43728**B**

Developments during the NIST competition

- Fiat-Shamir shown secure in the QROM - SOFIA becomes superfluous
- MQDSS proven secure in the QROM - still, sizes are huge
- several approaches that drastically improve the signature size
- Mudfish [Beullens, Eurocrypt '20]
  - Idea to reduce soundness error by introducing a preprocessing phase with a trusted **Helper**
  - And then have a regular Σ-protocol (satisfies completeness, special soundness, HVZK)
  - Takes inspiration from SOFIA and MPC-in-the-head [Katz, Kolesnikov, Wang, '18]
- MEDS, ALTEQ - Fiat-Shamir Goldreich-Micali-Wigderson scheme based on variants of Isomorphism of Polynomials
- MQOM - Fiat-Shamir based on MPC-in-the-head paradigm

Zero-Knowledge Interactive Proof of knowledge from Equivalence Problems

[Goldreich–Micali–Wigderson '91]:

Let ϕ be an isomorphism s.t. $\mathcal{O}_1 = \phi(\mathcal{O}_0)$.

Given $\mathcal{O}_0, \mathcal{O}_1$, the prover $\mathcal{P}$ wants to prove to the verifier $\mathcal{V}$ knowledge of ϕ without revealing any information about it

$$\begin{array}{c} \mathcal{O}_0 \\ \Big\downarrow{\scriptstyle \phi} \\ \mathcal{O}_1 \end{array}$$

$\mathcal{P}(\mathcal{O}_0, \mathcal{O}_1, \phi)$	$\mathcal{V}(\mathcal{O}_0, \mathcal{O}_1)$

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$$\begin{tikzcd} \mathcal{O}_0 \arrow[r, "\phi_0"] \arrow[d, dashed, "\phi"'] & \mathcal{O}' \\ \mathcal{O}_1 & \end{tikzcd}$$

$\mathcal{P}(\mathcal{O}_0, \mathcal{O}_1, \phi)$	$\mathcal{V}(\mathcal{O}_0, \mathcal{O}_1)$

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\hline
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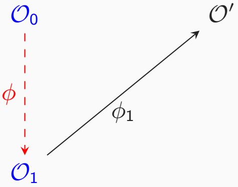

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Interesting concrete hard equivalence problems?

Generic hard equivalence problem $\mathrm{EQ}(\mathcal{O}_0, \mathcal{O}_1)$:
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- **Matrix code equivalence** - with Tung Chou, Ruben Niederhagen, Edoardo Persichetti, Tovohery Hajatiana Randrianarisoa, Lars Ran, Krijn Reijnders, Monika Trimoska , 2022
- ...

The Matrix Code Equivalence Problem

Matrix code - a subspace of $\mathcal{M}_{m\times n}(\mathbb{F}_q)$ of dimension k endowed with **rank metric**

$$d(\mathbf{A}, \mathbf{B}) = \mathrm{Rank}(\mathbf{A} - \mathbf{B})$$

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Matrix Code Equivalence (MCE) problem [Berger,2003]

Input: Two k-dimensional matrix codes $\mathcal{C},\mathcal{D} \subset \mathcal{M}_{m,n}(q)$

Question: Find – if any – $\mathbf{A} \in \mathrm{GL}_m(q), \mathbf{B} \in \mathrm{GL}_n(q)$ (an isometry) s.t. for all $\mathbf{C} \in \mathcal{C}$, it holds that

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Related problems

- *Matrix Codes Right (Left) Equivalence* problem (MCRE) – **A** (**B**) is trivial

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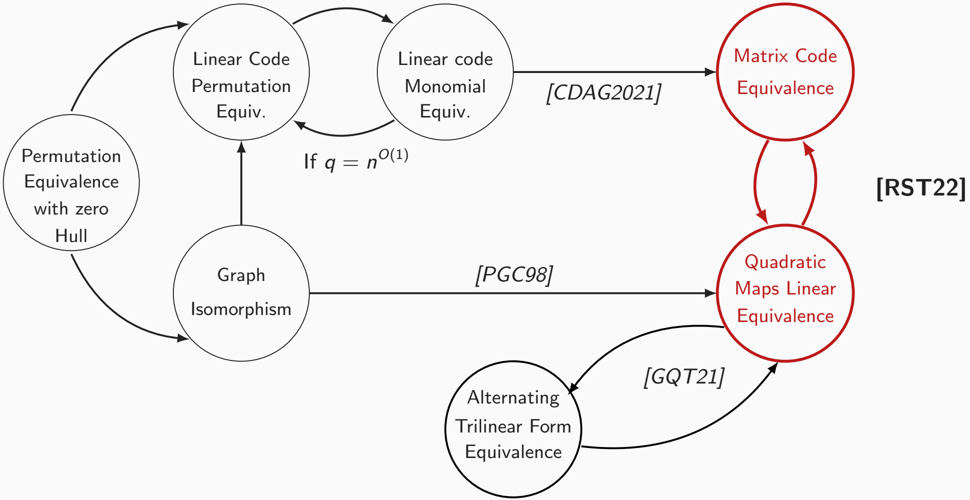
Related problems

- *Matrix Codes Right (Left) Equivalence* problem (MCRE) – $\mathbf{A}$ ($\mathbf{B}$) is trivial
- $\mathbb{F}_{q^m}$*-linear codes* – MCE reduces to MCRE

Known results - relations to other problems



Quadratic Maps Linear Equivalence (QMLE) problem

Introduced by Patarin 1996 as Isomorphism of Polynomials (IP) problem for building an identification scheme and FS signature!

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Input: Two k-tuples of quadratic multivariate polynomials $\mathcal{F},\ \mathcal{P} \in \mathbb{F}_q[x_1, \ldots, x_n]^k$

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- **homogenous version** hQMLE - **hard**
- inhomogenous version - easy (heuristic result [FP06])

Alternating trilinear form equivalence problem (ATFE)

Alternating trilinear form: $\phi(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \sum_{1 \leqslant i < j < s \leqslant n} c_{ijs} \begin{vmatrix} x_i & y_i & z_i \\ x_j & y_j & z_j \\ x_s & y_s & z_s \end{vmatrix}$ where $c_{ijs} \in \mathbb{F}_q$.

- Can be stored using $\binom{n}{3}$ entries: one for each c_{ijs} coefficient

Alternating trilinear form equivalence (ATFE) problem

Input: Two alternating trilinear forms $\phi, \psi \in \mathbb{F}_q[\mathbf{x}, \mathbf{y}, \mathbf{z}]$.

Question: Find – if any – invertible $\mathbf{A}$ such that $\psi(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \phi(\mathbf{A}\mathbf{x}, \mathbf{A}\mathbf{y}, \mathbf{A}\mathbf{z})$.

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- Used by Tang et al. '22 to build a signature scheme with competitive signature sizes
- Shown to be equivalent to hQMLE (Grochow et al. '21)

MCE, QMLE and ATFE look very similar!

Matrix codes:

MCE:

- matrix codes of rectangular matrices

MCE, QMLE **and** ATFE **look very similar!**

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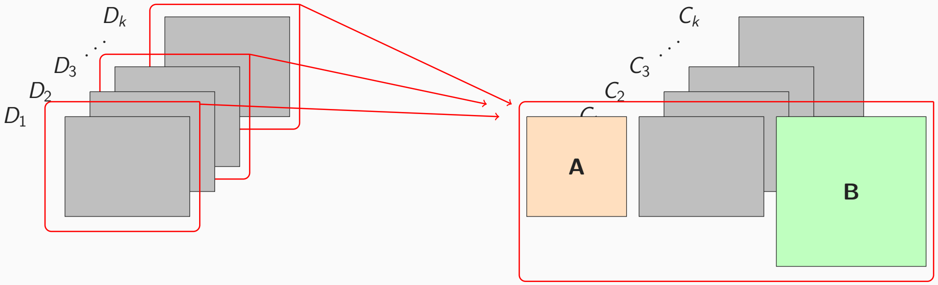

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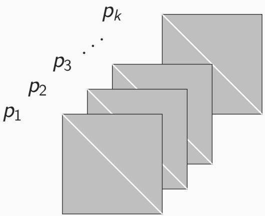
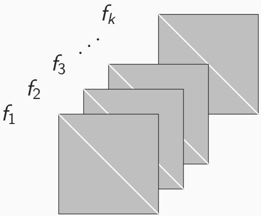
Systems of polynomials in matrix representation:





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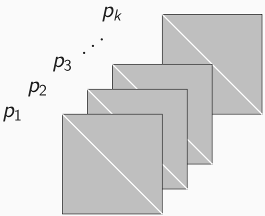

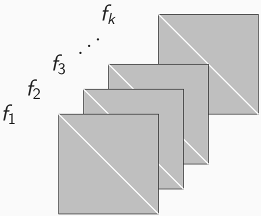

hQMLE:

- matrix codes of symmetric matrices

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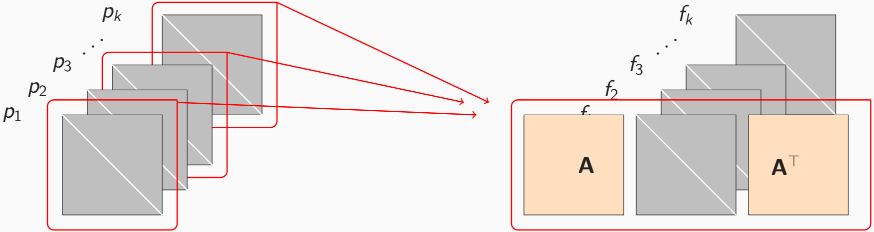

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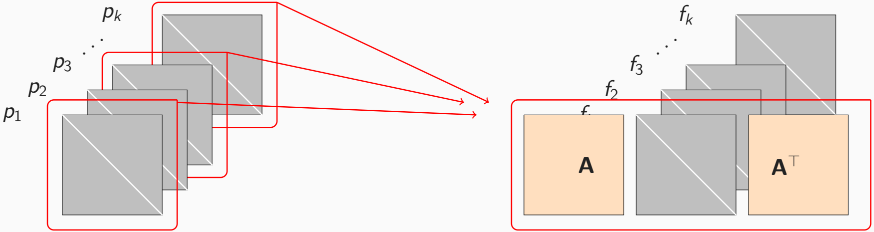

hQMLE:

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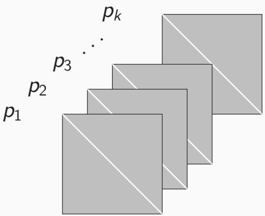
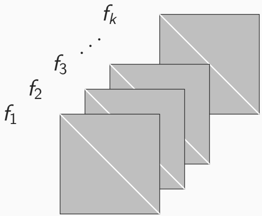
Trilinear forms in matrix representation:





ATFE:

- matrix codes of skew symmetric matrices

We will come back to cryptanalysis of this problem!

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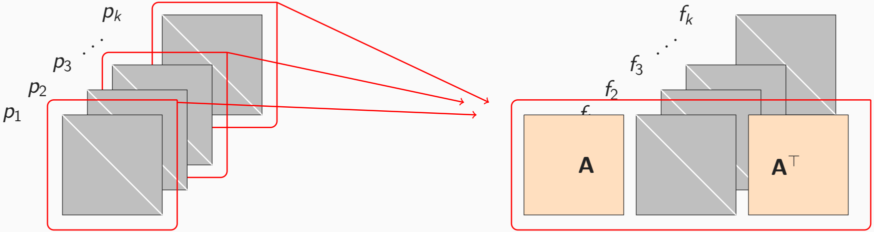

ATFE:

- matrix codes of skew symmetric matrices
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The basic protocol is not very eficient

$$
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$\mathsf{resp} \leftarrow \psi_{\mathsf{ch}_1}, \psi_{\mathsf{ch}_2}, \ldots, \psi_{\mathsf{ch}_r}$	$\xrightarrow{\mathsf{resp}}$	
		$\mathcal{O}' \stackrel{?}{=} \psi_{\mathsf{ch}_1}(\mathcal{O}_{\mathsf{ch}_1})$
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- For security of λ bits, **needs to be repeated $r = \lambda$ times!**

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- **Challenge space is of size 2** $\Rightarrow$ Soundness error is $1/2$
- For security of λ bits, **needs to be repeated $r = \lambda$ times!**
- $\Rightarrow$ Signature contains λ isometries (from λ rounds)

The basic protocol is not very eficient

$$\begin{tikzcd} \mathcal{O}_0 \arrow[r, "\psi_0"] \arrow[d, dashed, "\phi"'] & \mathcal{O}' \\ \mathcal{O}_1 \arrow[ur, "\psi_1"'] & \end{tikzcd}$$

$\mathcal{P}(\mathcal{O}_0, \mathcal{O}_1, \phi)$		$\mathcal{V}(\mathcal{O}_0, \mathcal{O}_1)$
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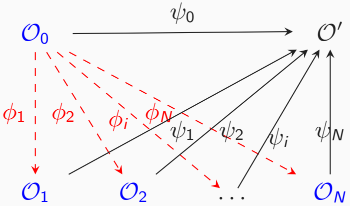

$\mathcal{P}(\mathcal{O}_0, \dots, \mathcal{O}_N, \phi_1, \dots, \phi_N)$		$\mathcal{V}(\mathcal{O}_0, \dots, \mathcal{O}_N)$
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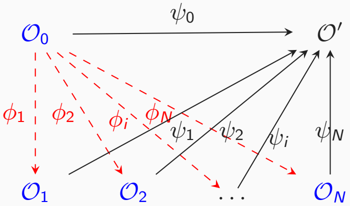
Optimization 1: Make the challenge space bigger (Multiple public keys)



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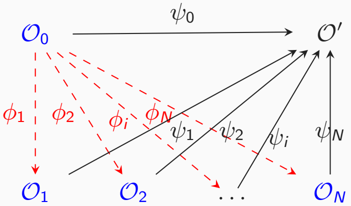

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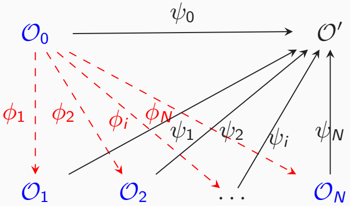

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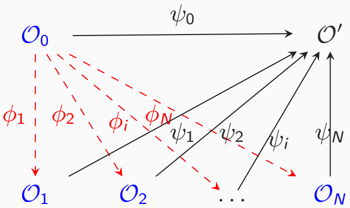

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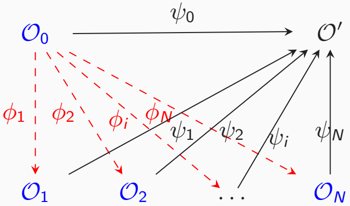

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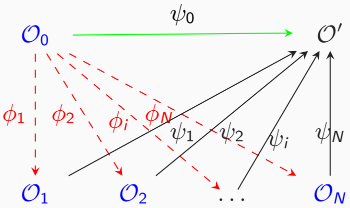

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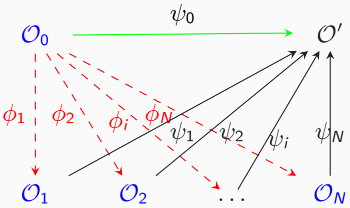

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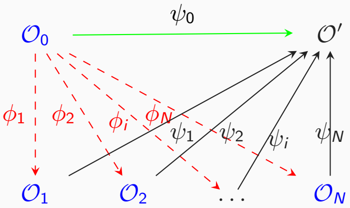

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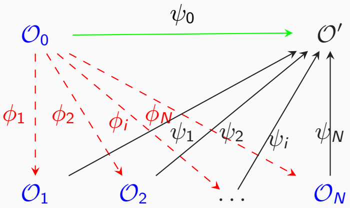

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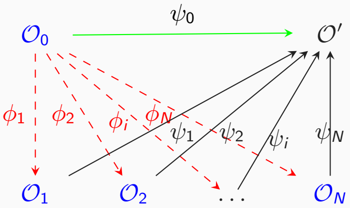

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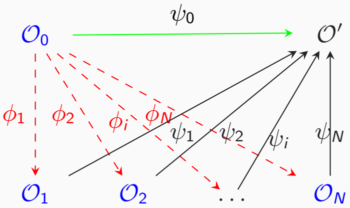

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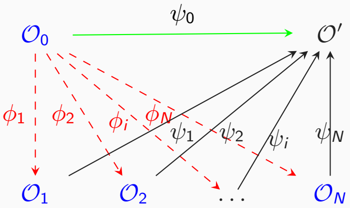

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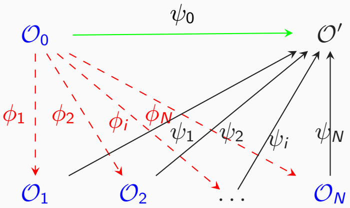

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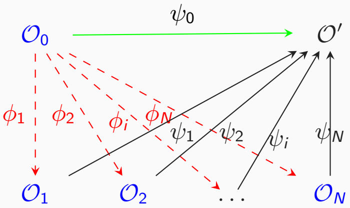

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So how better are these schemes in terms of performance?

For example, MEDS:

MEDS	q	n ($\approx m$)	r	pk (bytes)	Signature (bytes)
level 1	4093	25	144	21595	5456
level 2	4093	34	208	55520	10786
level 3	4093	44	272	122000	21052

Important to note:

- QMLE/MCE still not so well understood
- Not NP-hard but likely still hard
- Secure practical parameters significantly changed in the last few years, as our understanding improved

Solving matrix code equivalence problems

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Several approaches:

- Graph-based techniques

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- Leon-like approach (graph-based+algebraic)

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    $\Rightarrow \mathcal{P}'(\mathbf{x}) = \mathcal{P}(\mathbf{x}+\mathbf{a}), \mathcal{F}'(\mathbf{x}) = \mathcal{F}(\mathbf{x}+\mathbf{b})$ **is easy instance!**
  - **Problem is reduced to finding the collision**

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For $(\mathbf{C}^{(1)}, \dots, \mathbf{C}^{(k)})$ and $(\mathbf{D}^{(1)}, \dots, \mathbf{D}^{(k)})$ bases of $\mathcal{C}$ and $\mathcal{D}$, find invertible $\mathbf{A}, \mathbf{B}$ and $\mathbf{T} = (t_{ij})$ s.t.:

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- **Directly:** bilinear system of knm eqns in $m^2 + n^2 + k^2$ vars (bilinear $a_{ij} b_{i'j'}$, linear t_{rs})

Algebraic modelling of MCE - the straightforward way

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- For $m = n = k$, $\alpha = 2$ is enough $\rightarrow$ complexity is $\mathcal{O}(q^{2n} n^6)$

Improved Algebraic modelling of MCE

Make use of coding theory!

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- $\mathbf{G}$ and $\mathbf{G}'$ – the $k \times mn$ generator matrices of $\mathcal{C}$ and $\mathcal{D}$
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- Solve system with Bilinear XL for example
- Dimension of code – crucial for complexity
  - smallest for $k = mn/2$, and grows as k reduces or grows

Further improvments of the algebraic modelling of MCE

Trilinear form view:

$$\mathcal{C}(\mathbf{Ax}, \mathbf{By}, \mathbf{Tz}) = \mathcal{D}(\mathbf{x}, \mathbf{y}, \mathbf{z})$$

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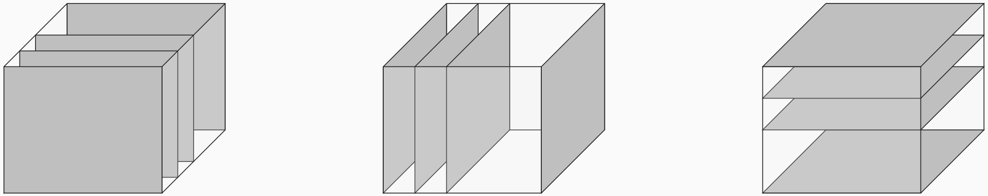

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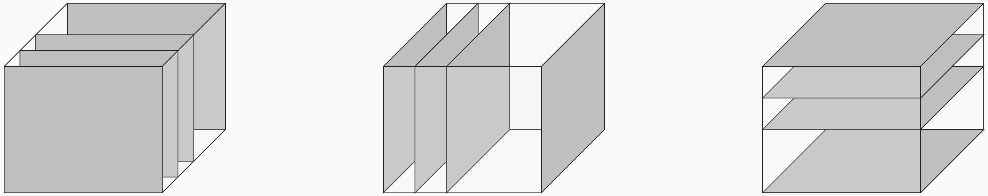

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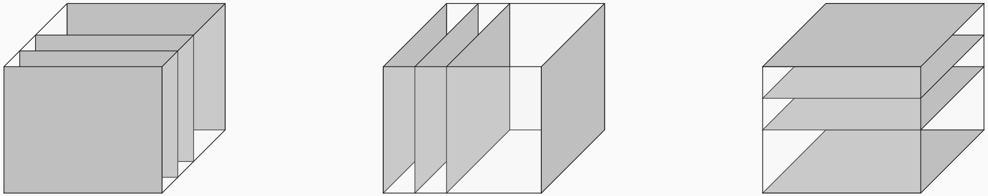

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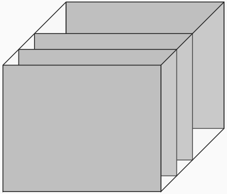

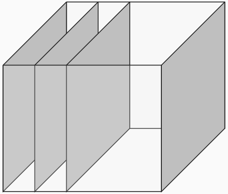

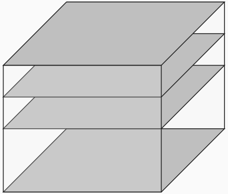

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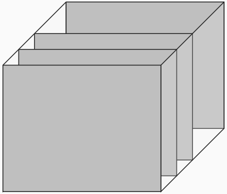

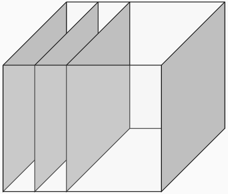

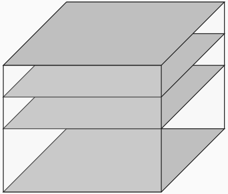

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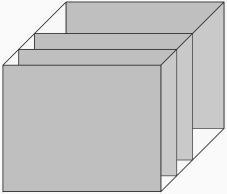

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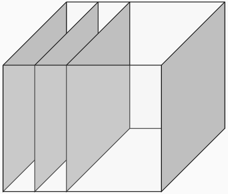

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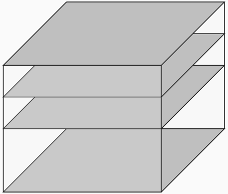

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$n = m = k$	plain	improved
14	169	148
22	255	218
30	349	299

Summary of the course

- **Monday - Designs**
  - General
  - Classic designs
- **Tuesday - Design and general MQ solving techniques**
  - Key size optimization techniques
  - Algorithms for solving the MQ problem
- **Wednesday - Cryptanalysis**
  - MinRank
  - Equivalent keys attacks
- **Thursday - Cryptanalysis**
  - Attacks on UOV
- **Friday - Provably secure designs**
  - Fiat-Shamir signatures MQDSS, SOFIA, MEDS

Thank you for listening!

And attending this course!